Ricci Flow on Complete Noncompact Manifolds

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Abstract

In this thesis, we will discuss some results which are related to Ricci flow on a complete noncompact manifold with possibly unbounded curvature.

In the first part, we will discuss the result on the short time existence of Ricci h-flow on a complete noncompact manifold M by M. Simon [16]. The result is as follows: if the metric g_0 is $1 + \epsilon(n)$ fair to the metric h which has bounded curvature, then Ricci h-flow exists on M with initial metric g_0. In order to obtain the result, one first consider a compact domain D and obtain a solution to the initial and boundary value problem. After some derivation of local a priori estimates for its derivatives. a solution of the Ricci h-flow on the whole manifold will be constructed.

In the second part, we will study the result on the short time existence of Ricci flow on a complete noncompact manifold with positive complex sectional curvature by E. Cabezas-Rivas and B. Wilking [10]. By considering the doubling of convex sets obtained in the Cheeger-Gromoll exhaustion and solving the initial value problem for the Ricci flow on these closed manifolds, a sequence of closed Ricci flows on a fixed time interval with positive complex sectional curvature is obtained. A curvature estimate around the soul point is then derived. This enables one to obtain a limit solution on the whole manifold.

摘要

在這篇畢業論文中，我們將討論兩個關於在無曲率上限的非緊且完備黎曼流形上的里奇流的結果。

在第一部分中，我們會先探討里奇 $\hbar$ 流在非緊且完備的黎曼流形上的存在結果。這結果是由 M.Simon[16]在 2002 年證明的。他證明了在任意一個非緊且完備的 n 維黎曼流形上，而流形上有一度量 $\hbar$，且 $\hbar$ 的曲率為有限數值。若果流形上有另一度量g_0，而g_0並無曲率上限條件。只要其滿足與 $\hbar, \epsilon(n)$ fair 的條件，則以 g_0 為初度量的里奇 $\hbar$ 流 $g(t)$ 存在，並且當 $t > 0$ 時，其曲率為有限。他首先在流形上的任一緊集上求出一里奇 $\hbar$ 流的 Dirichlet 解。透過得出對其導數的先驗估計，可以從而得出一個在其流形上的里奇 $\hbar$ 流解。

而在第二部分中，我們將探討 Wilking 和 CABEZAS-RIVAS [10]的結果。他們證明了在任意一個非緊且完備的 n 維黎曼流形中，若其度量 g 的複截面曲率為正數，則以 g 為初度量的里奇流 $g(t)$ 存在。他們利用了 Cheeger-Gromoll 的凸窮舉集，並且考慮以其凸集建構的二重覆蓋。在該緊集上求一里奇流解，而且其解的時間區間與該凸集無關。透過估計其曲率上限，我們可以求得在整個黎曼流形上以 g 為初度量的里奇流解。

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Contents

Chapter 1

Introduction

In this thesis, we consider a evolution equation to deform the metric on any n-dimensional Riemannian manifold (M, g_0):

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} &= -2R_{ij}, \\ g(0) &= g_0, \end{cases} \tag{1.1}$$

where R_{ij} is the Ricci curvature of M.

In [26], Hamilton introduced this geometric flow (called Ricci flow) and proved that every compact three manifold with positive Ricci curvature admitting a metric of constant positive sectional curvature. The Ricci flow has then been proved to be very useful in the research of differential geometry. The first important thing which we have to concern is its short time existence. In the case where M is a compact Riemannian manifold, Hamilton [26] proved that for any C^∞ initial metric g_0, Ricci flow equation has a unique solution for a short time using Nash-Moser inverse function theorem. Later on, Dennis DeTurck [9] gave a elegant proof on the existence and uniqueness of Ricci flow on closed manifold in which he modified the Ricci flow into a nonlinear parabolic equation. Therefore the short time existence problem is solved in the compact case. But the complete noncompact case is more difficult.

In [9], in stead of considering the Ricci flow equation, DeTurck considered the

Ricci-Deturck flow which is a strictly parabolic system. The Ricci-Deturck flow is the solution of the following evolution equation:

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} &= -2R_{ij} +^t \nabla_i W_j +^t \nabla_j W_i, \\ g(0) &= g_0, \end{cases} \tag{1.2}$$

where $W = W(g)$ is defined by $W_j = g_{jk} g^{pq} ({}^g\Gamma^k_{pq} -^h \Gamma^k_{pq})$ and h is a fixed background metric on M. After solving the Ricci-Deturck flow on M, he pulled it back through a diffeomorphism and obtain a Ricci flow on M.

Following the idea of DeTurck, Shi [32] considered open manifold (complete noncompact) (M, g_0) with bounded curvature. He evolved (M, g_0) by the same evolution equation as in (1.2) with $h = g_0$. He showed that the Ricci-DeTurck flow exists on M which can be pulled back to a Ricci flow on M. Thus, the short time existence problem is solved if (M, g) has bounded curvature.

Due to the work by Shi, we attempt to remove any restriction on curvature bounds for open manifolds. For non-compact 2-manifolds (possibly incomplete and with unbounded curvature), this was settled by Giesen and Topping in [11], using the idea from [22]. But for $n \geq 3$, it is difficult to imagine how to construct a solution of Ricci flow. And it seems to be necessary to requre more informations about the curvature.

In chapter 2, we will present the result by M. Simon in [16] in which he considered (M, g_0) with unbounded curvature but g_0 is C^0-close to a background metric h with bounded curvature in the following sense.

Definition 1.0.1. *Let M be a complete manifold and g a C^0 metric , and $\delta \in [1, +\infty)$, a given constant. A metric h is said to be δ-fair to g, if h is C^∞ and there exists a constant k_0 such that*

$$\sup_{x \in M} |Riem(h)(x)|_h = k_0 < +\infty, \tag{1.3}$$

and

$$\frac{1}{\delta} h(p) \leq g(p) \leq \delta h(p), \quad \forall p \in M. \tag{1.4}$$

He evolved (M, g_0) by the same equation in (1.2) with h being the fixed background metric. He called it the Ricci h-flow. He proved that if the initial metric g_0 is $1 + \delta$ fair to h for sufficiently small $\delta(n) > 0$, then the Ricci h-flow exists on M for short time $t \in [0, T(n, k_0)]$. Later on, he proved that the Ricci h-flow can be pulled back to a Ricci flow on $\mathbb{R}^n$ with h being the flat metric, see [19].

In chapter 3, we will discuss the result by B. Wilking and E. Cabezas-Rivas in [10] in which they considered open manifolds with nonnegative complex sectional curvature.

Here, we explain the meaning of complex sectional curvature on (M, g). Consider its complexified tangent bundle $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$, we extend the curvature tensor Rm and its metric g at p to $\mathbb{C}-$multilinear maps. The complex sectional curvature of a 2-dimensional complex subspace σ of $T_p^{\mathbb{C}}M$ is defined by

$$K^{\mathbb{C}}(\sigma) = Rm(u, v, \bar{u}, \bar{v}),$$

where u and v form an unitary basis for σ. And we say that M has nonnegative complex sectional curvature if $K^{\mathbb{C}} \geq 0$.

In [10], B. Wilking and E. Cabezas-Rivas constructed a Ricci flow solution on open manifold (M, g) with $K_g^{\mathbb{C}} \geq 0$ without any assumptions on the curvature upper bounds. The result is based on the work of Cheeger, Gromoll in [13]. They proved that for open manifolds with $K_g \geq 0$, it admits an exhaustion by convex set C_l. It enables us to construct a Ricci flow with nonnegative complex sectional curvature on the closed manifold formed by gluing two copies of C_l along the common boundary and whose initial metric is the natural singular metric on the double. By passing to limit, the following theorem can be obtained.

Theorem 1.0.2. *Let (M^n, g) be an open manifold with nonnegative (and possibly unbounded) complex sectional curvature. Then there exists a constant T depending on n and g such that (1.1) has a smooth solution on the interval $[0, T]$, with $g(0) = g$ and with $g(t)$ having nonnegative complex sectional curvature.*

B. Wilking and E. Cabezas-Rivas also showed that the maximal existence time of Ricci flow can be estimated from below by the supremum of volume of balls in (M, g).

Corollary 1.0.3. *In each dimension n, there is a universal constant $\epsilon(n) > 0$ such that for each complete manifold (M^n, g) with $K_g^{\mathbb{C}} \geq 0$, the following holds: If we put*

$$\tau = \epsilon(n) \cdot \sup \left\{ \frac{vol_g(B_g(p,r))}{r^{n-2}} : p \in M, r > 0 \right\} \in (0, \infty],$$

then any complete maximal solution of Ricci flow $(M, g(t)), t \in [0, T)$ with $K_{g(t)}^{\mathbb{C}} \geq 0$ and $g(0) = g$ satisfies $\tau \leq T$.

Chapter 2

h Ricci flow

2.1 Introduction to h-Ricci flow

In this chapter, we study the Ricci h-flow which is a variant of the Ricci-Deturck flow. For a Riemannian manifold (M, g_0) with a fixed background metric h, we define the h-flow with initial data g_0 by

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} &= -2R_{ij} +^t \nabla_i W_j +^t \nabla_j W_i, \\ g(0) &= g_0, \end{cases} \tag{2.1}$$

where the time dependent 1-form $W = W(g(t))$ is defined by

$$W_j = g_{jk} g^{pq} (\Gamma^k_{pq} - {}^h\Gamma^k_{pq}).$$

Here, R_{ij} is the Ricci curvature of $g(t)$ and ${}^t\nabla$ stands for the connection induced by the metric $g(t)$. Γ^i_{jk} and ${}^h\Gamma^i_{jk}$ are the Christoffel symbols of the metrics $g(t)$ and h respectively.

Throughout this chapter, we will denote $\overline{Rm}$ and $\overline{\nabla}$ as the curvature tensor of h and the connection induced by the metric h respectively.

Lemma 2.1.1. *In local coordinate, h-flow solves the evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} &= g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{ij} - g^{\alpha\beta}g_{ip}h^{pq}\overline{R}_{j\alpha q\beta} - g^{\alpha\beta}g_{jp}h^{pq}\overline{R}_{i\alpha q\beta}\\ &\quad + \frac{1}{2}g^{\alpha\beta}g^{pq}(\overline{\nabla}_i g_{p\alpha}\overline{\nabla}_j g_{q\beta} + 2\overline{\nabla}_\alpha g_{jp}\overline{\nabla}_q g_{i\beta} - 2\overline{\nabla}_\alpha g_{jp}\overline{\nabla}_\beta g_{iq}\\ &\quad - 2\overline{\nabla}_j g_{p\alpha}\overline{\nabla}_\beta g_{iq} - 2\overline{\nabla}_i g_{p\alpha}\overline{\nabla}_\beta g_{jq}).\end{aligned}$$

where $\overline{\nabla}$ is the connection induced by the metric h, $\overline{R} = Riem(h)$. In particular, the evolution equation is a strictly parabolic system.

Proof. See Lemma 2.1 in [32]. □

We now show the relations between Ricci flow and the h-flow. Suppose $g(t), t \in [0, T]$ solves h-flow, we consider a 1-parameter family of maps $\phi_t(x) : M \to M$ by the equation

$$\begin{cases} \frac{\partial}{\partial t}\phi_t(p) &= -W(\phi_t(p), t)\\ \phi_0 &= id, \end{cases} \tag{2.2}$$

If the equation (2.2) has a smooth solution ϕ_t on $M \times [0, T]$ and remains diffeomorphism on $[0, T]$. One can observe that the family of metrics $\tilde{g}(t) \doteq \phi_t^* g(t)$ is a solution to the Ricci flow. We compute as follows.

$$\begin{aligned}\frac{\partial}{\partial t}(\phi_t^* g(t)) &= \frac{\partial}{\partial s}|_{s=0}(\phi_{s+t}^* g(t+s))\\ &= \phi_t^*(\frac{\partial}{\partial t}g(t)) + \frac{\partial}{\partial s}|_{s=0}(\phi_{s+t}^*(g(t)))\\ &= \phi_t^*(-2Ric(g(t) + \mathcal{L}_{W(t)}g(t)) + \frac{\partial}{\partial s}|_{s=0}[(\phi_t^{-1}\circ\phi_{t+s})^*\phi_t^*(g(t))]\\ &= -2Ric(\phi_t^* g(t)) + \phi_t^*(\mathcal{L}_{W(t)}g(t)) - \mathcal{L}_{(\phi_t^{-1})_* W(t)}\phi_t^* g(t)\\ &= -2Ric(\phi_t^* g(t)).\end{aligned}$$

where $\mathcal{L}_X g$ refers to the Lie derivative of metric g in the direction of X.

In this chapter, we will prove the short time existence of h-flow in the case that g_0 is smooth and $(1+\epsilon)$-fair to the metric h with ϵ sufficiently small. We define the fairness as follows.

Definition 2.1.2. *Let M be a complete manifold and g a C^0 metric , and $\delta \in [1, +\infty)$, a given constant. A metric h is said to be δ-fair to g, if h is C^∞ and there exists a constant k_0 such that*

$$\sup_{x \in M} |Riem(h)(x)|_h = k_0 < +\infty, \tag{2.3}$$

and

$$\frac{1}{\delta} h(p) \le g(p) \le \delta h(p), \quad \forall p \in M. \tag{2.4}$$

Remark : By the result of Shi, Theorem 1.1 in [32], if h_0 is a smooth Riemannian metric with bounded curvature k_0, then there exists a constant $T = T(n, k_0) > 0$ such that the Ricci flow $h(t)$ with initial metric h_0 exists on M for $0 \le t \le T$. It satisfies the following estimates. For each $m \in \mathbb{N}$, there exists constants $c_m = c(n, m, k_0) > 0$ such that

$$\sup_{x \in M} |\nabla^m R_{ijkl}(x,t)|^2 \le \frac{c_m}{t^m}, \quad 0 \le t \le T.$$

From the Ricci flow equation ,

$$\frac{\partial}{\partial t} h(t) = -2Ric(h(t)), \quad 0 \le t \le T,$$

It follows that

$$|\frac{\partial}{\partial t} h_{ij}|^2 \le 4|R_{ij}|^2 \le 4n^2 c_0, \quad 0 \le t \le T,$$

This implies

$$e^{-2n\sqrt{c_0}t} h(x,0) \le h_{ij}(x,t) \le e^{2n\sqrt{c_0}t} h_{ij}(x,0), \forall x \in M, t \in [0,T].$$

For all $\epsilon > 0$, we can find $t_0 = t_0(\epsilon, n, k_0) > 0$ small enough such that the new metric $h'(x) = h(x, t_0)$ satisfies

$$\frac{1}{1+\epsilon} h_0(x) \le h'(x) \le (1+\epsilon) h_0(x), \quad \forall x \in M$$

and

$$\sup_{x \in M} |{}^{h'}\nabla^m Riem(h')(x)|^2_{h'} \le \frac{c_m}{t_0^m} < +\infty.$$

So, if h is $(1+\delta)$-fair to a metric g, then we can replace h by another smooth metric h' which is $(1+2\delta)$ fair to g ,and

$$\sup_{x \in M} |{}^{h'}\nabla^m Riem(h')(x)|^2_{h'} = k_j < +\infty,$$

So we can assume that h always fulfills such estimates.

2.2 Evolution equations of derivatives of g

In this section, we state the evolution equations for the derivatives of the h-flow.

Lemma 2.2.1. *Suppose $g(t)$ is a solution of h-flow and h is $1+\epsilon(n)$ fair to $g(t)$, then in local coordinate, we have*

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\nabla}^m g_{ab} =& g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla}^m g_{ab} + \sum_{i+j+k=m; i,j,k\le m} \overline{\nabla}^i g^{-1} * \overline{\nabla}^j g * \overline{\nabla}^k Riem(h) \\ &+ \sum_{i+j+k+l=m; i,j,k,l\le m} \overline{\nabla}^i g^{-1} * \overline{\nabla}^j g^{-1} * \overline{\nabla}^{k+1} g * \overline{\nabla}^{l+1} g, \quad \forall m \in \mathbb{N}\end{aligned}$$

where here $T * S$ (T and S are tensors) refers to some trace with respect to the metric h which results in a tensor of the appropriate type.

Proof. We prove it by induction on m.

When $m = 1$, differentiate the evolution equation in Lemma (2.1.1).

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\nabla} g_{ab} &= \overline{\nabla}(g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{ab}) - \overline{\nabla}(g^{\alpha\beta}g_{ap}h^{pq}\overline{R}_{b\alpha q\beta} + g^{\alpha\beta}g_{bp}h^{pq}\overline{R}_{a\alpha q\beta}) \\ &\quad + \overline{\nabla}\left[\frac{1}{2}g^{\alpha\beta}g^{pq}(\overline{\nabla}_a g_{p\alpha}\overline{\nabla}_b g_{q\beta} + 2\overline{\nabla}_\alpha g_{bp}\overline{\nabla}_q g_{a\beta} - 2\overline{\nabla}_\alpha g_{bp}\overline{\nabla}_\beta g_{aq} - 4\overline{\nabla}_a g_{p\alpha}\overline{\nabla}_\beta g_{bq})\right] \\ &= g^{\alpha\beta}\overline{\nabla}\,\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{ab} + \overline{\nabla} g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{ab} + \overline{\nabla}(g^{-1} * g * h * \overline{Rm}) \\ &\quad + \overline{\nabla}(g^{-1} * g^{-1} * \overline{\nabla} g * \overline{\nabla} g) \\ &= g^{\alpha\beta}\overline{\nabla}\,\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{ab} + \overline{\nabla} g^{-1} * \overline{\nabla}^2 g + \overline{\nabla} g^{-1} * g * \overline{Rm} + g^{-1} * \overline{\nabla} g * \overline{Rm} \\ &\quad + g^{-1} * g * \overline{\nabla}\,\overline{Rm} + g^{-1} * \overline{\nabla} g^{-1} * \overline{\nabla} g * \overline{\nabla} g + g^{-1} * g^{-1} * \overline{\nabla}^2 g * \overline{\nabla} g.\end{aligned}$$

By differentiating the equation $g^{ij}g_{jk} = \delta^i_k$, we have $\overline{\nabla}_i g^{jk} = -g^{jp}g^{kq}\overline{\nabla}_i g_{pq}$. It gives

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\nabla} g_{ab} &= g^{\alpha\beta}\overline{\nabla}\,\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{ab} + \overline{\nabla} g^{-1} * g * \overline{Rm} + g^{-1} * \overline{\nabla} g * \overline{Rm} + g^{-1} * g * \overline{\nabla}\,\overline{Rm} \\ &\quad + g^{-1} * \overline{\nabla} g^{-1} * \overline{\nabla} g * \overline{\nabla} g + g^{-1} * g^{-1} * \overline{\nabla}^2 g * \overline{\nabla} g.\end{aligned}$$

By using Ricci identity on the first term, one can deduce that

$$\begin{aligned}g^{\alpha\beta}\overline{\nabla}\,\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{ab} &= g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}\,\overline{\nabla}_\beta g_{ab} + g^{-1} * \overline{Rm} * \overline{\nabla} g \\ &= g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla} g_{ab} + g^{-1} * \overline{\nabla}(\overline{Rm} * g) + g^{-1} * \overline{Rm} * \overline{\nabla} g \\ &= g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla} g_{ab} + g^{-1} * \overline{\nabla} g * \overline{Rm} + g^{-1} * g * \overline{\nabla}\,\overline{Rm}\end{aligned}$$

So, we have

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\nabla} g_{ab} &= g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla} g_{ab} + \left[\overline{\nabla} g^{-1} * g * \overline{Rm} + g^{-1} * \overline{\nabla} g * \overline{Rm} + g^{-1} * g * \overline{\nabla}\,\overline{Rm}\right] \\ &\quad + \left[g^{-1} * \overline{\nabla} g^{-1} * \overline{\nabla} g * \overline{\nabla} g + g^{-1} * g^{-1} * \overline{\nabla}^2 g * \overline{\nabla} g\right].\end{aligned}$$

The case of $m = 1$ is true. Suppose it is true for $m = p$. That is

$$\begin{aligned}\frac{\partial}{\partial t}\overline{\nabla}^p g_{ab} =& g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla}^p g_{ab} + \sum_{i+j+k=p;i,j,k\le p}\overline{\nabla}^i g^{-1} * \overline{\nabla}^j g * \overline{\nabla}^k\overline{Rm} \\ &+ \sum_{i+j+k+l=p;i,j,k,l\le p}\overline{\nabla}^i g^{-1} * \overline{\nabla}^j g^{-1} * \overline{\nabla}^{k+1} g * \overline{\nabla}^{l+1} g.\end{aligned}$$

Differentiate the above equation. We obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\overline{\nabla}_c\overline{\nabla}^p g_{ab} &= \overline{\nabla}_c(g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla}^p g_{ab}) + \overline{\nabla}_c\left[\sum_{i+j+k=p,i,j,k\le p}\overline{\nabla}^i g^{-1} * \overline{\nabla}^j g * \overline{\nabla}^k\overline{Rm}\right] \\
&\quad + \overline{\nabla}_c\left[\sum_{i+j+k+l=p,i,j,k,l\le p}\overline{\nabla}^i g^{-1} * \overline{\nabla}^j g^{-1} * \overline{\nabla}^{k+1} g * \overline{\nabla}^{l+1} g\right] \\
&= \overline{\nabla}_c g^{\alpha\beta}\ \overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla}^p g_{ab} + g^{\alpha\beta}\overline{\nabla}_c\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla}^p g_{ab} \\
&\quad + \left[\sum_{i+j+k=p+1,i,j,k\le p+1}\overline{\nabla}^i g^{-1} * \overline{\nabla}^j g * \overline{\nabla}^k\overline{Rm}\right] \\
&\quad + \left[\sum_{i+j+k+l=p+1;i,j,k,l\le p+1}\overline{\nabla}^i g^{-1} * \overline{\nabla}^j g^{-1} * \overline{\nabla}^{k+1} g * \overline{\nabla}^{l+1} g\right].
\end{aligned}$$

Apply Ricci identity again on the second term, we obtain

$$g^{\alpha\beta}\overline{\nabla}_c\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla}^p g_{ab} = g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla}_c\overline{\nabla}^p g_{ab} + \overline{Rm} * \overline{\nabla}^{p+1} g + \overline{\nabla}\,\overline{Rm} * \overline{\nabla}^p g.$$

Together with the fact that $\overline{\nabla}_c g^{jk} = -g^{jp}g^{kq}\overline{\nabla}_c g_{pq}$, we get the desired equation.

$$\begin{aligned}
\frac{\partial}{\partial t}\overline{\nabla}^{p+1} g_{ab} &= g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\overline{\nabla}^{p+1} g_{ab} + \left[\sum_{i+j+k=p+1,i,j,k\le p+1}\overline{\nabla}^i g^{-1} * \overline{\nabla}^j g * \overline{\nabla}^k\overline{Rm}\right] \\
&\quad + \left[\sum_{i+j+k+l=p+1;i,j,k,l\le p+1}\overline{\nabla}^i g^{-1} * \overline{\nabla}^j g^{-1} * \overline{\nabla}^{k+1} g * \overline{\nabla}^{l+1} g\right].
\end{aligned}$$

By mathematical induction, result follows. □

Thus, we have the evolution equation of the norm of derivatives of $g(t)$.

Lemma 2.2.2. *Suppose $g(t)$ is a solution of h-flow and $g(t)$ is $1+\epsilon(n)$ fair to h, then in local coordinate, we have*

$$\begin{aligned}
\frac{\partial}{\partial t}|\overline{\nabla}^m g|^2 &\le g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta|\overline{\nabla}^m g|^2 - 2g^{\alpha\beta}\langle\overline{\nabla}_\alpha\overline{\nabla}^m g, \overline{\nabla}_\beta\overline{\nabla}^m g\rangle \\
&\quad + c(m,n,h)\sum_{i+j\le m;i,j\le m}|\overline{\nabla}^i g||\overline{\nabla}^j g||\overline{\nabla}^m g| \\
&\quad + c(m,n,h)\sum_{i+j+k+l=m+2;i,j,k,l\le m+1}|\overline{\nabla}^i g||\overline{\nabla}^j g||\overline{\nabla}^k g||\overline{\nabla}^l g||\overline{\nabla}^m g|
\end{aligned}$$

Here, the norm of tensor is with respect to the metric h.

Proof.

$$(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)|\overline{\nabla}^m g|^2 = \langle(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)\overline{\nabla}^m g, \overline{\nabla}^m g\rangle - 2g^{\alpha\beta}\langle\overline{\nabla}_\alpha\overline{\nabla}^m g, \overline{\nabla}_\beta\overline{\nabla}^m g\rangle,$$

Substitute the result of Lemma (2.2.1) into the above equation, together with the fairness assumption, we obtain the result. □

2.3 Zero order estimate of h-flow.

Before we prove the existence of Dirichlet solution, we first need some a priori estimate of the solution. In this section, we will give zero estimate on the Dirichlet solution.

Lemma 2.3.1. *Let D be a compact region in M. Let g_0 be a $C^\infty(D)$ metric and h, a metric on M which satisfies*

$$g_0 \geq \frac{h}{1+\delta}.$$

Let $g(t)$, $t \in [0,T]$ be a $C^\infty(D \times [0,T])$ solution to the h-flow with Dirichlet boundary conditions $g|_{\partial D}(\cdot,t) = g_0$, $g(0) = g_0$. Then for every $\sigma > 0$, there exists an $S = S(n, k_0, \delta, \sigma) > 0$ such that

$$g(t) \geq \frac{h}{(1+\delta)(1+\sigma)}, \forall t \in [0,T] \cap [0,S].$$

Proof. We define a function $\phi : M \times [0,T] \to \mathbb{R}$ by $\phi(x,t) = g^{i_1j_1}h_{j_1i_2}..g^{i_mj_m}h_{j_mi_1}$ (m is a integer to be chosen). At a fixed point $p \in M$, we may always find a local coordinates around p, such that, $h_{ij}(p) = \delta_{ij}$ and $g_{ij} = \delta_{ij}\lambda_i(p)$. In this local coordinate, we have $\phi(x,t) = \sum_{i=1}^n \frac{1}{\lambda_i^m}$.

Noted that it satisfies

$$\sup_{(x,t)\in D\times\{0\}\cup\partial D\times[0,T]} \phi(x,t) \leq n(1+\delta)^m.$$

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x,t) &= g^{i_1 j_1} h_{j_1 i_2} \ldots h_{j_{k-1} i_k} (\frac{\partial}{\partial t} g^{i_k j_k}) h_{j_k i_{k+1}} \ldots g^{i_m j_m} h_{j_m i_1} \\ &= g^{i_1 j_1} h_{j_1 i_2} \ldots h_{j_{k-1} i_k} (-g^{i_k q} g^{p j_k} \frac{\partial}{\partial t} g_{pq}) h_{j_k i_{k+1}} \ldots h_{j_m i_1}.\end{aligned}$$

In our preferred coordinate,

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x,t) &= -\frac{m}{\lambda_i^{m+1}} \frac{\partial}{\partial t} g_{ii} \\ &= -\frac{m}{\lambda_i^{m+1}} \{ g^{\alpha\beta} \overline{\nabla}_\alpha \overline{\nabla}_\beta g_{ii} - \frac{2\lambda_i}{\lambda_\alpha} \overline{R}_{i\alpha i\alpha} \\ &\quad + \frac{1}{2\lambda_\alpha \lambda_p} [(\overline{\nabla}_i g_{p\alpha})^2 + 2\overline{\nabla}_\alpha g_{ip} \overline{\nabla}_p g_{i\alpha} - 2(\overline{\nabla}_\alpha g_{ip})^2 - 4\overline{\nabla}_i g_{p\alpha} \overline{\nabla}_\alpha g_{ip}] \} \\ &= \frac{-m}{\lambda_i^{m+1}\lambda_\alpha} \overline{\nabla}_\alpha \overline{\nabla}_\alpha g_{ii} + \frac{2m}{\lambda_i^m \lambda_\alpha} \overline{R}_{i\alpha i\alpha} \\ &\quad - \frac{m}{2\lambda_i^{m+1} \lambda_\alpha \lambda_p} [(\overline{\nabla}_i g_{p\alpha})^2 + 2\overline{\nabla}_\alpha g_{ip} \overline{\nabla}_p g_{i\alpha} - 2(\overline{\nabla}_\alpha g_{ip})^2 - 4\overline{\nabla}_i g_{p\alpha} \overline{\nabla}_\alpha g_{ip}].\end{aligned}$$

$$\overline{\nabla}_\beta \phi = g^{i_1 j_1} \ldots (\overline{\nabla}_\beta g^{i_k j_k}) \ldots h_{j_m i_1}.$$

$$\begin{aligned}& g^{\alpha\beta} \overline{\nabla}_\alpha \overline{\nabla}_\beta \phi \\ &= g^{\alpha\beta} g^{i_1 j_1} \ldots (\overline{\nabla}_\beta g^{i_k j_k}) \ldots (\overline{\nabla}_\beta g^{i_l j_l}) \ldots h_{j_m i_1} + g^{\alpha\beta} g^{i_1 j_1} \ldots (\overline{\nabla}_\alpha \overline{\nabla}_\beta g^{i_k j_k}) \ldots h_{j_m i_1} \\ &= \frac{m}{\lambda_\alpha} (\overline{\nabla}_\alpha g^{ij})^2 (\sum_{k=0}^{m-2} \frac{1}{\lambda_i^{m-2-k} \lambda_j^k}) + \frac{m}{\lambda_i^{m-1}\lambda_\alpha} \overline{\nabla}_\alpha \overline{\nabla}_\alpha g^{ii} \\ &= \frac{m}{\lambda_\alpha} (\overline{\nabla}_\alpha g^{ij})^2 (\sum_{k=0}^{m-2} \frac{1}{\lambda_i^{m-2-k} \lambda_j^k}) + \frac{m}{\lambda_i^{m-1}\lambda_\alpha} (-g^{ip} g^{iq} \overline{\nabla}_\alpha \overline{\nabla}_\alpha g_{pq} + 2g^{ip} g^{il} g^{qk} \overline{\nabla}_\alpha g_{pq} \overline{\nabla}_\alpha g_{lk}) \\ &= \frac{m}{\lambda_\alpha} (\overline{\nabla}_\alpha g^{ij})^2 (\sum_{k=0}^{m-2} \frac{1}{\lambda_i^{m-2-k} \lambda_j^k}) + \frac{m}{\lambda_i^{m-1}\lambda_\alpha} [\frac{-1}{\lambda_i^2} \overline{\nabla}_\alpha \overline{\nabla}_\alpha g_{ii} + \frac{2}{\lambda_i^2 \lambda_\alpha} (\overline{\nabla}_\alpha g_{ip})^2].\end{aligned}$$

$$
\begin{aligned}
&(\partial_t - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)\phi \\
= &-\frac{m}{2\lambda_i^{m+1}\lambda_\alpha\lambda_p}[(\overline{\nabla}_i g_{p\alpha})^2 + 2\overline{\nabla}_\alpha g_{ip}\overline{\nabla}_p g_{i\alpha} - 2(\overline{\nabla}_\alpha g_{ip})^2 - 4\overline{\nabla}_i g_{p\alpha}\overline{\nabla}_\alpha g_{ip}] \\
&-\frac{m}{\lambda_\alpha}(\overline{\nabla}_\alpha g^{ij})^2(\sum_{k=0}^{m-2}\frac{1}{\lambda_i^{m-2-k}\lambda_j^k}) + \frac{2m}{\lambda_i^m\lambda_\alpha}\overline{R}_{i\alpha i\alpha} - \frac{2m}{\lambda_i^{m+1}\lambda_\alpha\lambda_p}(\overline{\nabla}_\alpha g_{ip})^2) \\
\leq &-\frac{m}{2\lambda_i^{i+1}\lambda_\alpha\lambda_p}[(\overline{\nabla}_i g_{p\alpha})^2 + 2\overline{\nabla}_\alpha g_{ip}\overline{\nabla}_p g_{i\alpha} - 2(\overline{\nabla}_\alpha g_{ip})^2 \\
&- 4\overline{\nabla}_i g_{p\alpha}\overline{\nabla}_\alpha g_{ip} + 4(\overline{\nabla}_\alpha g_{ip})^2] + \frac{2m}{\lambda_i^m\lambda_\alpha}\overline{R}_{i\alpha i\alpha} \\
\leq &-\frac{m}{2\lambda_i^{i+1}\lambda_\alpha\lambda_p}[(\overline{\nabla}_i g_{p\alpha})^2 + 2\overline{\nabla}_\alpha g_{ip}\overline{\nabla}_p g_{i\alpha} - 4\overline{\nabla}_i g_{p\alpha}\overline{\nabla}_\alpha g_{ip} + 2(\overline{\nabla}_\alpha g_{ip})^2] \\
&+ \frac{2mk_0}{\lambda_i^m\lambda_\alpha} \\
\leq &-\frac{m}{2\lambda_i^{i+1}\lambda_\alpha\lambda_p}[(\overline{\nabla}_i g_{p\alpha})^2 + 2\overline{\nabla}_\alpha g_{ip}\overline{\nabla}_p g_{i\alpha} - 2\overline{\nabla}_i g_{p\alpha}\overline{\nabla}_\alpha g_{ip} \\
&- 2\overline{\nabla}_i g_{p\alpha}\overline{\nabla}_p g_{i\alpha} + (\overline{\nabla}_\alpha g_{ip})^2 + (\overline{\nabla}_p g_{i\alpha})^2] + \frac{2mk_0}{\lambda_i^m\lambda_\alpha} \\
\leq &\frac{2mk_0}{\lambda_i^m\lambda_\alpha} - \frac{m}{2\lambda_i^{i+1}\lambda_\alpha\lambda_p}(-\overline{\nabla}_i g_{p\alpha} + \overline{\nabla}_\alpha g_{ip} + \overline{\nabla}_p g_{i\alpha})^2 \\
\leq &C\phi^{1+\frac{1}{m}}
\end{aligned}
$$

where the constant $C = C(m, k_0, n) = 2mnk_0$.

We now define $f : M \times [0, T] \to \mathbb{R}$ by $f = \phi^{\frac{-1}{m}}$,

$$
\begin{aligned}
\partial_t f &= -\frac{1}{m}\phi^{\frac{-1-m}{m}}\partial_t\phi \\
g^{ij}\overline{\nabla}_i\overline{\nabla}_j f &= -\frac{1}{m}g^{ij}\phi^{\frac{-1-m}{m}}\overline{\nabla}_i\overline{\nabla}_j\phi + \frac{m+1}{m^2}\phi^{\frac{-1-2m}{m}}g^{ij}\overline{\nabla}_i\phi\overline{\nabla}_j\phi
\end{aligned}
$$

$$
\begin{aligned}
&(\partial_t - g^{ij}\overline{\nabla}_i\overline{\nabla}_j)f \\
= &-\frac{1}{m}\phi^{\frac{-1-m}{m}}(\partial_t - g^{ij}\overline{\nabla}_i\overline{\nabla}_j)\phi + \frac{m+1}{m^2}\phi^{\frac{-1-2m}{m}}g^{ij}\overline{\nabla}_i\phi\overline{\nabla}_j\phi \\
\geq &-\frac{C}{m} + (m+1)\phi^{\frac{1}{m}}g^{ij}\overline{\nabla}_i f\overline{\nabla}_j f.
\end{aligned}
$$

By parabolic maximum principle, this implies that $\forall (x, t) \in D \times [0, T]$

$$
\begin{aligned}
f + \frac{Ct}{m} &\geq \inf_{D\times\{0\}\cup\partial D\times[0,T]} f \\
&\geq \frac{1}{\sup_{D\times\{0\}\cup\partial D\times[0,T]}(\sum_{i=1}^n \frac{1}{\lambda_i^m})^{\frac{1}{m}}} \geq \frac{1}{n^{\frac{1}{m}}(1+\delta)}
\end{aligned}
$$

And hence, we have on $D \times [0,T]$

$$\phi(x,t) = \sum_{i=1}^{n} \frac{1}{\lambda_i^m} \leq [\frac{1}{n^{\frac{1}{m}}(1+\delta)} - 2nk_0 t]^{-m},$$

It implies that, for any $i \in \{1,2,...n\}$, $(x,t) \in D \times [0,T]$,

$$\lambda_i(x,t) \geq \frac{1}{n^{\frac{1}{m}}(1+\delta)} - 2nk_0 t.$$

For any given $\sigma > 0$, choose $m \in \mathbb{N}$ large enough so that $(1+\frac{\sigma}{2}) > n^{\frac{1}{m}}$ and choose $S(n,k_0,\sigma,\delta) = \dfrac{1+\sigma}{2nk_0(1+\delta)(1+\sigma/2)} > 0$, then we have

$$\lambda_i(x,t) \geq \frac{1}{(1+\delta)(1+\sigma)}, \forall (x,t) \in D \times [0,S].$$

□

We wish to obtain bounds from above for $g(t)$ in terms of h.

Lemma 2.3.2. *There exists a constant $\tilde{\epsilon} = \tilde{\epsilon}(n) > 0$ such that the followings hold:*
Let D be a compact region in M, and g_0 be a $C^\infty(D)$ metric and h a metric on M which satisfies $h \leq g_0 \leq (1+\delta)h$ ($\delta < \tilde{\epsilon}$). Let $g(t)$, $t \in [0,T]$ be a $C^\infty(D \times [0,T])$ solution to the h-flow with Dirichlet boundary conditions $g|_{\partial D}(\cdot,t) = g_0(\cdot), g(0) = g_0$. Then there exists $S = S(n,k_0,\delta) > 0$ such that

$$g(t) \leq (1+2\delta)h, \forall t \in [0,T] \cap [0,S].$$

Proof. Let $1 > \tilde{\epsilon} = \tilde{\epsilon}(n) > 0$ be a constant such that

$$\log \frac{1+2\delta}{1+\delta} < \frac{\log 2n}{1168 n^5}, \quad \forall \delta < \tilde{\epsilon}. \tag{2.5}$$

Choose $m \in \mathbb{N}$ such that

$$\frac{\log 2n}{\log(1+2\delta) - \log(1+\delta)} \leq m \leq \frac{2\log 2n}{\log(1+2\delta) - \log(1+\delta)}. \tag{2.6}$$

Let $\alpha(m) = \dfrac{1}{m}$. By the previous theorem, there exists an $S = S(n, k_0, \alpha) > 0$ such that

$$g(t) \geq \frac{h}{1+\alpha}, \forall t \in [0,T] \cap [0,S].$$

We define a function $G : M \times [0,T] \to \mathbb{R}$ by $G = h^{i_1 j_1} g_{j_1 i_2} ... h^{i_m j_m} g_{j_m i_1}$.

From the fact that h is $1+\delta$ fair to g_0, we can see that

$$n \leq G(x,0) \leq n(1+\delta)^m$$

And for $(1+\delta)^m - \frac{1}{2n} G > 0$, we define $F = \dfrac{1}{(1+\delta)^m - \frac{1}{2n} G}$.

Clearly, we know that

$$(1+\delta)^m - \frac{1}{2n} G(x,0) \geq \frac{1}{2}(1+\delta)^m, \tag{2.7}$$

and hence $F(x,0) < \infty$, is well defined at $t = 0$.

Since D is compact, and $g(x,t)$ is a priori smooth, there is $T' \in [0,T] \cap [0,S]$, such that $F(x,t)$ is well defined for all $t \in [0,T')$, and if $\sup_{D \times [0,T')} F(x,t) < \infty$, then $[0,T'] = [0,T] \cap [0,S]$. Since F is well defined on $[0,T')$, we see that $(1+\delta)^m - \frac{1}{2n} G(x,t) > 0$ for all $t \in [0,T')$. In our preferred coordinate, it implies

$$\lambda_i \leq (2n)^{\frac{1}{m}} (1+\delta), \forall i \in \{1,2,...n\}, \forall t \in [0,T').$$

Now we evaluate the evolution equation for F.

$$\begin{aligned}
\frac{\partial}{\partial t} G &= h^{i_1 j_1} g_{j_1 i_2} ... h^{i_l j_l} (\frac{\partial}{\partial t} g_{j_l i_{l+1}}) h^{i_{l+1} j_{l+1}} ... g_{j_m i_1} \\
&= m \lambda_i^{m-1} \frac{\partial}{\partial t} g_{ii} \\
&= m \lambda_i^{m-1} \{ g^{\alpha\beta} \overline{\nabla}_\alpha \overline{\nabla}_\beta g_{ii} - \frac{2\lambda_i}{\lambda_\alpha} \overline{R}_{i\alpha i\alpha} \\
&\quad + \frac{1}{2\lambda_\alpha \lambda_p} [(\overline{\nabla}_i g_{p\alpha})^2 + 2\overline{\nabla}_\alpha g_{ip} \overline{\nabla}_p g_{i\alpha} - 2(\overline{\nabla}_\alpha g_{ip})^2 - 4\overline{\nabla}_i g_{p\alpha} \overline{\nabla}_\alpha g_{ip}] \}.
\end{aligned}$$

$$\overline{\nabla}_\beta G = h^{i_1j_1} g_{j_1i_2}...(\overline{\nabla}_\beta g_{j_l i_{l+1}})...g_{j_m i_1}.$$

$$g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta G$$
$$= h^{i_1j_1} g_{j_1i_2}...(g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{j_l i_{l+1}})...g_{j_m i_1} + g^{\alpha\beta}h^{i_1j_1} g_{j_1i_2}...(\overline{\nabla}_\alpha g_{j_k i_{k+1}})...(\overline{\nabla}_\beta g_{j_l i_{l+1}})...g_{j_m i_1}$$
$$= m\lambda_i^{m-1}(g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{ii}) + \frac{m}{\lambda_\alpha}(\overline{\nabla}_\alpha g_{ij})^2(\sum_{k=0}^{m-2}\lambda_i^{m-2-k}\lambda_j^k).$$

$$(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)G$$
$$= m\lambda_i^{m-1}\big\{g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{ii} - \frac{2\lambda_i}{\lambda_\alpha}\overline{R}_{i\alpha i\alpha}$$
$$+ \frac{1}{2\lambda_\alpha\lambda_p}[(\overline{\nabla}_i g_{p\alpha})^2 + 2\overline{\nabla}_\alpha g_{ip}\overline{\nabla}_p g_{i\alpha} - 2(\overline{\nabla}_\alpha g_{ip})^2 - 4\overline{\nabla}_i g_{p\alpha}\overline{\nabla}_\alpha g_{ip}]\big\}$$
$$- m\lambda_i^{m-1}(g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta g_{ii}) - \frac{m}{\lambda_\alpha}(\overline{\nabla}_\alpha g_{ij})^2(\sum_{k=0}^{m-2}\lambda_i^{m-2-k}\lambda_j^k)$$
$$= -\frac{2m\lambda_i^m}{\lambda_\alpha}\overline{R}_{i\alpha i\alpha} - \frac{m}{\lambda_\alpha}(\overline{\nabla}_\alpha g_{ij})^2(\sum_{k=0}^{m-2}\lambda_i^{m-2-k}\lambda_j^k)$$
$$+ \frac{m\lambda_i^{m-1}}{2\lambda_\alpha\lambda_p}[(\overline{\nabla}_i g_{p\alpha})^2 + 2\overline{\nabla}_\alpha g_{ip}\overline{\nabla}_p g_{i\alpha} - 2(\overline{\nabla}_\alpha g_{ip})^2 - 4\overline{\nabla}_i g_{p\alpha}\overline{\nabla}_\alpha g_{ip}].$$
$$\leq \frac{2mk_0}{\lambda_\alpha}G - \frac{m}{\lambda_\alpha}(\overline{\nabla}_\alpha g_{ij})^2(\sum_{k=0}^{m-2}\lambda_i^{m-2-k}\lambda_j^k) + \frac{m\lambda_i^{m-1}}{2\lambda_\alpha\lambda_p}[3\overline{\nabla}_\alpha g_{ip}\overline{\nabla}_p g_{i\alpha} + 2(\overline{\nabla}_i g_{\alpha p})^2]$$

Using the fact that ,

$$\frac{1}{1+\alpha} \leq \lambda_i \leq (2n)^{\frac{1}{m}}(1+\delta), \quad on \quad [0,T').$$

We conclude that

$$(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)G$$
$$\leq 2mnk_0(1+\alpha)G + m(1+\alpha)^2|\overline{\nabla}g|^2\big[6n(1+\alpha)(1+\delta)^m - \frac{m-1}{(2n)^{1/m}(1+\delta)(1+\alpha)^m}\big].$$

By using (2.5), (2.6), it implies that

$$\frac{1}{2}\log(1+\delta) < \log\frac{1+2\delta}{1+\delta}$$
$$73n(1+\delta)^m < 73n(16n^4) = 1168n^5 < \frac{\log 2n}{\log\frac{1+2\delta}{1+\delta}} \leq m$$

Thus, we conclude that

$$1+6n(2n)^{\frac{1}{m}}(1+\frac{1}{m+1})^{m+1}(1+\delta)^{m+1}<1+6n(2)(3)(2)(1+\delta)^m<73(1+\delta)^m\leq m.$$

$$6n(1+\alpha)(1+\delta)^m<\frac{m-1}{(2n)^{1/m}(1+\delta)(1+\alpha)^m}$$

So we get

$$(\frac{\partial}{\partial t}-g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)G\leq 2mnk_0(1+\alpha)G.$$

$$\begin{aligned}(\frac{\partial}{\partial t}-g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)F&=\frac{F^2}{2n}(\frac{\partial}{\partial t}-g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)G-\frac{F}{n}g^{\alpha\beta}\overline{\nabla}_\beta G\overline{\nabla}_\alpha F\\&\leq 2mnk_0(1+\alpha)(1+\delta)^mF^2-\frac{F^3}{2n^2\lambda_\alpha}(\overline{\nabla}_\alpha G)^2\\&\leq 2mnk_0(1+\alpha)(1+\delta)^mF^2.\end{aligned}$$

By parabolic maximum principle and (2.7) , we obtain

$$F(\cdot,t)\leq\frac{a}{1-bt},\forall t\in[0,T'). \tag{2.8}$$

where $a=\sup_{x\in D}F(x,0)$, $b=2mnk_0(1+\alpha)(1+\delta)^m a$.

Without loss of generality, we assume that $S\leq\frac{1}{2b}$, which implies that $bt\leq\frac{1}{2}$, $\forall t\in[0,T')$. By mean of (2.7), we obtain

$$F(\cdot,t)\leq\frac{4}{(1+\delta)^m},\forall t\in[0,T'),$$

And hence $T'=\min(S,T)$.

Combining with (2.6), we get

$$\lambda_i\leq(2n)^{\frac{1}{m}}(1+\delta)\leq(1+2\delta),\quad\forall t\in[0,T]\cap[0,S].$$

□

Combing the above two lemmas, we can conclude the following theorem.

Theorem 2.3.3. *There exists a $\tilde{\epsilon}(n) > 0$ such that the followings hold:*
Let D be a compact region in M, and g_0 be a $C^\infty(D)$ metric and h, a metric on M which satisfies $\frac{1}{1+\delta}h \le g_0 \le (1+\delta)h$ where $\delta(\delta+2) \le \tilde{\epsilon}$. Let $g(t), t \in [0,T]$ be a $C^\infty(D \times [0,T])$ smooth solution to h flow with Dirichlet boundary conditions $g|_{\partial D}(\cdot, t) = g_0(\cdot), g(0) = g_0$. Then there exists $S = S(\delta, n, k_0)$ such that

$$\frac{1}{(1+3\delta)}h \le g(t) \le (1+3\delta)h, \quad \forall t \in [0,T] \cap [0,S].$$

Proof. First note that if $g(t)$ is a solution to h-flow, then $(1+\delta)g(\frac{t}{1+\delta})$ is also a solution to h-flow, with initial data $(1+\delta)g_0$. Let $\tilde{g}(t) = (1+\delta)g(\frac{t}{1+\delta})$, $\tilde{g}_0 = \tilde{g}(0) = (1+\delta)g_0$ satisfies $h \le \tilde{g}(0) \le (1+\delta)^2 h$. From the Lemma 2.3.2, we may find an $S = S(n, k_0, \delta) > 0$ so that

$$\tilde{g}(t) \le (1 + 4\delta + 2\delta^2)h, \quad \forall t \in [0, (1+\delta)S] \cap [0, (1+\delta)T].$$

which will imply

$$g(t) \le \frac{1+4\delta+2\delta^2}{1+\delta}h \le (1+3\delta)h, \quad \forall t \in [0,S] \cap [0,T].$$

By Lemma 2.3.1, we can find $S = S(n, k_0, \delta) > 0$ such that

$$\frac{1}{(1+3\delta)}h \le g(t) \le (1+3\delta)h, \quad \forall t \in [0,S] \cap [0,T].$$

. □

2.4 A priori interior estimates for the gradient and higher derivatives of g.

In order to prove the existence of Dirichlet solutions on arbitrary compact set D, we need some a priori estimate on the derivatives of $g(t)$. After that, we wish to let D go to infinity on M to get a limit solution on M. To do this, we need to control $g_{ij}(x,t)$ locally. In this section, we will give some a priori estimates on the gradient and higher derivatives of $g(t)$.

Lemma 2.4.1. *Let $g(t)$,$t \in [0,S]$ be a $C^\infty(D \times [0,S])$ solution to the h flow with Dirichlet boundary conditions $g|_{\partial D}(\cdot,t) = g_0(\cdot), g(0) = g_0$, for some h which is $1+\epsilon(n)$ fair to $g(t)$, $\forall t \in [0,S]$ ($\epsilon(n)$ to be specified in the proof below). Then*

$$\sup_{x \in D} |\overline{\nabla} g(x,t)|^2 \le c(n,h,D,g_0|_D),\ \forall t \in [0,S].$$

The norm is with respect to the background metric h.

Proof. Let $\phi : M \times [0,S] \to \mathbb{R}$ by,

$$\phi(x,t) = g_{j_1 i_1} h^{i_1 j_2} g_{j_2 i_2} h^{i_2 j_3} ... g_{j_m i_m} h^{i_m j_1}.$$

$m = m(n)$ is a integer to be chosen. Choose $\epsilon(n) = \dfrac{1}{m(n)}$.
We may choose a coordinates at point p such that $h_{ij}(p) = \delta_{ij}$, $g(p)_{ij} = \lambda_i(p,t)\delta_{ij}$. In this coordinate, $\phi(x,t) = \lambda_1^m + ...\lambda_n^m$. Since $(1-\epsilon)h(x) \le g(x,t) \le (1+\epsilon)h(x)$, we have

$$1-\epsilon \le \lambda_i \le 1+\epsilon, \quad \forall (x,t) \in M \times [0,S].$$

Compute the evolution equation of $\phi(x,t)$ as before, we will obtain the following equation.

$$\begin{aligned}
(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha \overline{\nabla}_\beta)\phi \le & -\frac{m}{\lambda_\alpha}(\overline{\nabla}_\alpha g_{ij})^2 (\sum_{k=0}^{m-2} \lambda_i^{m-2-k}\lambda_j^k) + \frac{2mk_0}{\lambda_\alpha}\phi \\
& + \frac{m\lambda_i^{m-1}}{2\lambda_\alpha \lambda_p}[3\overline{\nabla}_\alpha g_{ip} \overline{\nabla}_p g_{i\alpha} + 2(\overline{\nabla}_i g_{\alpha p})^2] \\
\le & \frac{2mnk_0}{1-\epsilon}(1+\epsilon)^m \\
& + \left[\frac{3m(1+\epsilon)^{m-1}}{(1-\epsilon)^2} - \frac{m(m-1)(1-\epsilon)^{m-2}}{1+\epsilon}\right] |\overline{\nabla} g|^2 \\
\le & 12mnk_0 - \frac{m^2}{8}|\overline{\nabla} g|^2.
\end{aligned}$$

Now define $\psi : M \times [0,S] \to \mathbb{R}$ by $\psi = (\phi + a)|\overline{\nabla} g|^2$, $a(n) > 0$ is a constant to be chosen.

By Lemma (2.2.2), we have

$$
\begin{aligned}
(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)|\overline{\nabla} g|^2 &= -2g^{\alpha\beta}\langle\overline{\nabla}_\alpha\overline{\nabla} g, \overline{\nabla}_\beta\overline{\nabla} g\rangle + g^{-1} * \overline{Rm} * \overline{\nabla} g * \overline{\nabla} g \\
&\quad + \overline{\nabla}\,\overline{Rm} * g * g^{-1} * \overline{\nabla} g + g^{-1} * g^{-1} * \overline{\nabla} g * \overline{\nabla} g * \overline{\nabla}\,\overline{\nabla} g \\
&\quad + g^{-1} * g^{-1} * g * \overline{\nabla} g * \overline{\nabla} g * \overline{Rm} \\
&\le -2g^{\alpha\beta}\langle\overline{\nabla}_\alpha\overline{\nabla} g, \overline{\nabla}_\beta\overline{\nabla} g\rangle + C|\overline{\nabla} g|^2|\overline{\nabla}\overline{\nabla} g| \\
&\quad + C|\overline{\nabla} g|^2 + C|\overline{\nabla} g|^4 + C|\overline{\nabla} g| \qquad (\quad C = C(n, k_0, k_1, \epsilon) \quad) \\
&\le -|\overline{\nabla}^2 g|^2 + C|\overline{\nabla} g|^2|\overline{\nabla}^2 g| \\
&\quad + C|\overline{\nabla} g|^2 + C|\overline{\nabla} g|^4 + C|\overline{\nabla} g|. \qquad (\text{provided} \quad \epsilon(n) < 1.)
\end{aligned}
$$

By using Young's inequality, we can obtain

$$
(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)|\overline{\nabla} g|^2 \le -\frac{1}{2}|\overline{\nabla}^2 g|^2 + C_2|\overline{\nabla} g|^4 + C_3.
$$

Where C_2, C_3 depends on n, k_0, k_1 only.

$$
\begin{aligned}
&(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)\psi \\
&= (\phi + a)(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)|\overline{\nabla} g|^2 + |\overline{\nabla} g|^2(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)\phi - 2g^{ij}\overline{\nabla}_i\phi\overline{\nabla}_j|\overline{\nabla} g|^2 \\
&\le (\phi + a)(-\frac{1}{2}|\overline{\nabla}^2 g|^2 + C_2|\overline{\nabla} g|^4 + C_3) + |\overline{\nabla} g|^2(C_1 - \frac{m^2}{8}|\overline{\nabla} g|^2) - 2g^{ij}\overline{\nabla}_i\phi\overline{\nabla}_j|\overline{\nabla} g|^2 \\
&\le -\frac{\phi + a}{2}|\overline{\nabla}^2 g|^2 + C_2(\phi + a)|\overline{\nabla} g|^4 + C_3(\phi + a) + C_1|\overline{\nabla} g|^2 - \frac{m^2}{8}|\overline{\nabla} g|^4 - 2g^{ij}\overline{\nabla}_i\phi\overline{\nabla}_j|\overline{\nabla} g|^2
\end{aligned}
$$

$$
\begin{aligned}
-2g^{ij}\overline{\nabla}_i\phi\overline{\nabla}_j|\overline{\nabla} g|^2 &= -\frac{4m\lambda_k^{m-1}}{\lambda_i}\overline{\nabla}_i g_{kk}\langle\overline{\nabla}_j\overline{\nabla} g, \overline{\nabla} g\rangle \\
&\le \frac{C_4 m(1+\epsilon)^{m-1}}{1-\epsilon}|\overline{\nabla}^2 g||\overline{\nabla} g|^2 \qquad (\ C_4 = C_4(n)\ ) \\
&\le 2C_4 m|\overline{\nabla}^2 g||\overline{\nabla} g|^2 \\
&\le \frac{1}{2}(a + \phi)|\overline{\nabla}^2 g|^2 + \frac{2m^2(C_4)^2|\overline{\nabla} g|^4}{a + \phi}
\end{aligned}
$$

Choose $a(n)$ large enough so that $\dfrac{2m^2(C_4)^2}{a+\phi} \le \dfrac{m^2}{32}$. And also choose $m = m(n)$

large so that $C_2(\phi + a) \leq \frac{m}{16} \leq \frac{m^2}{16}$ and $\frac{1}{\tilde{\epsilon}(n)} < m$. Then we can deduce that

$$\begin{aligned}
(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)\psi &\leq -\frac{m^2}{32}|\overline{\nabla} g|^4 + C_1|\overline{\nabla} g|^2 + C_5(n, k_0, k_1) \\
&\leq -\frac{1}{2}\psi^2 + c_0(n, k_0, k_1) \\
&\leq c_0 \qquad (2.9)
\end{aligned}$$

From the maximum principle we obtain that

$$\sup_{D\times[0,S]} (\psi - c_0 t) \leq \sup_{\partial D\times[0,S]\cup D\times\{0\}} \psi \qquad (2.10)$$

Applying Lemma 3.1, VI, §3 in [18] to the evolution equation of h-flow, we get $\sup\limits_{\partial D\times[0,S]} |\overline{\nabla} g| \leq c(n, h, \partial D)$, in view of the a priori parabolicity. Together with (2.10), the result follows. □

Furthermore, we can also estimate the derivatives of $g(x,t)$, $\sup_{D\times[0,T]} |\overline{\nabla}^m g(x,t)|$, on compact set D by a constant depending only on m, n, h and $g_0|_D$. We summarize in the following theorem.

Lemma 2.4.2. *Let $g(t)$,$t \in [0,S]$ be a $C^\infty(D \times [0,S])$ solution to the h flow, for some h which is $1+\epsilon(n)$ fair to $g(t)$, $\forall t \in [0,S]$ ($\epsilon(n)$ to be specified in the proof of Lemma 2.4.1) with Dirichlet boundary conditions $g|_{\partial D}(\cdot,t) = g_0(\cdot), g(0) = g_0$. Then*

$$\sup_{x\in D} |\overline{\nabla}^m g(x,t)|^2 \leq \tilde{C}(m, n, h, D, g_0|_D),\ \forall t \in [0,S].$$

The norm is with respect to the background metric h.

Proof. By Lemma 2.4.1, we get that

$$\sup_{x\in D} |\overline{\nabla} g(x,t)|^2 \leq c(n, h, D, g_0|_D),\ \forall t \in [0,S].$$

Thus, if the h-flow is written in the form of

$$-\frac{\partial}{\partial t} g_{kl} + g^{ij}\overline{\nabla}_i\overline{\nabla}_j g_{kl} = f_{kl},$$

then f_{kl} is bounded uniformly by a constant depending only on $k_0, n, h, g_0|_D$ and D. Using Theorem 9.1, Ch IV, §9 in [18], we have $W_q^{2,1}$ estimates for $g(t)$ with

$$||g||_{W_q^{2,1}(D\times(0,S))} \leq c(k_0, n, h, g_0|_D, D, q) \text{ ,for all integer } q.$$

Thus, by Sobolev imbedding Theorem, we get that

$$||\overline{\nabla} g||_{C^{\alpha,\alpha/2}(D\times(0,\times S))} \leq c(k_0, n, h, g_0|_D, D, \alpha).$$

By a priori parabolicity, we can deduce that

$$\sup_{x\in D} |\overline{\nabla}^m g(x,t)|^2 \leq \tilde{C}(m, n, h, D, g_0|_D),\ \forall t \in [0, S].$$

□

Next, we give the local estimates of derivatives of $g(t)$ independent of the compact set D.

Lemma 2.4.3. *Let $g(t), t \in [0, S]$ be a $C^\infty(D \times [0, S])$ solution to the h flow, for some h which is $1+\epsilon(n)$ fair to $g(t)$, for all $t \in [0, S]$ (ϵ as in Lemma 2.4.1). Then*

$$\sup_{B(x_0,r)} |\overline{\nabla} g(x,t)|^2 \leq c(n, h, r)\frac{1}{t} \quad ,\forall t \in [0, S],$$

where $B(x_0, r)$ denotes a ball of radius r with centre x_0 with respect to the metric h. The norm is calculated with respect to metric h. The constant $c(n, h, r)$ decrease with the radius r.

Proof. By (2.9), we saw that the function $\psi(x,t) = (\phi(x,t) + a(n))|\overline{\nabla} g(x,t)|^2$ satisfies

$$(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)\psi \leq -\frac{1}{2}\psi^2 + c_0(n, k_0, k_1) \quad ,\forall (x,t) \in M \times [0, S] \qquad (2.11)$$

Define $f(x,t) = \psi(x,t)t$, f satisfies

$$(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)f \leq -\frac{1}{2}\frac{f^2}{t} + \frac{f}{t} + c_0 t \quad ,\forall (x,t) \in M \times [0, S].$$

For fixed $x_0 \in M$, as in [32] we can use the background metric h to construct time independent cut off function η satisfying

$$\eta(x) = 1, \qquad \forall x \in B(x_0, r), \tag{2.12}$$

$$\eta(x) = 0, \qquad \forall x \in M \backslash B(x_0, 2r), \tag{2.13}$$

$$0 \leq \eta(x) \leq 1 \quad , \forall x \in M, \tag{2.14}$$

$${}^h|\overline{\nabla}\eta|^2 \leq c_1 \left(\frac{1}{r}\right) \eta, \tag{2.15}$$

$$\overline{\nabla}_i \overline{\nabla}_j \eta \geq -c_2 \left(k_0, \frac{1}{r}\right) h_{ij}. \tag{2.16}$$

Note also that the function η is C^∞ almost everywhere and Lipschitz everywhere. We can mollify the function η and obtain a C^∞ function satisfying the same properties but for slightly different balls and slightly different constants. By choosing a new constant and new balls, we assume $\eta \in C^\infty(M)$.

Using the properties of η, we get

$$(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha \overline{\nabla}_\beta)(f\eta) \leq -\frac{1}{2}\frac{f^2\eta}{t} + \frac{f\eta}{t} + c_0 t\eta - f g^{\alpha\beta}\overline{\nabla}_\alpha \overline{\nabla}_\beta \eta - 2g^{\alpha\beta}\overline{\nabla}_\alpha f \overline{\nabla}_\beta \eta,$$

In this proof, we will use $c = c(n, h, r)$ to denote any constant which depends on n, h, r only. Assume (x_0, t_0) is an interior point of $B(x_0, 2r) \times [0, S]$ where $f\eta$ attains its maximum. Because of it , we get

$$\begin{aligned} -2g^{\alpha\beta}\overline{\nabla}_\alpha f \overline{\nabla}_\beta \eta &= -\frac{2}{\eta} g^{\alpha\beta}\overline{\nabla}_\alpha (f\eta)\overline{\nabla}_\beta \eta + \frac{2}{\eta} g^{\alpha\beta}\overline{\nabla}_\alpha \eta \overline{\nabla}_\beta \eta \\ &= \frac{2}{\eta} g^{\alpha\beta}\overline{\nabla}_\alpha \eta \overline{\nabla}_\beta \eta, \end{aligned}$$

at the point (x_0, t_0). Together with (2.15) , (2.16) ,it implies

$$-f g^{\alpha\beta}\overline{\nabla}_\alpha \overline{\nabla}_\beta \eta - 2g^{\alpha\beta}\overline{\nabla}_\alpha f \overline{\nabla}_\beta \eta \leq cf,$$

at (x_0, t_0). Consequently , we have

$$(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha \overline{\nabla}_\beta) f\eta \leq -\frac{1}{2}\frac{f^2\eta}{t} + \frac{f\eta}{t} + cf + c_0 S.$$

Since $f\eta$ attains maximum at (x_0, t_0) and $\eta \in [0,1]$, we can conclude that

$$\begin{aligned}
0 &\leq -\frac{1}{2t_0} f^2\eta(x_0,t_0) + \frac{f\eta(x_0,t_0)}{t_0} + cf(x_0,t_0) + c_0 S \\
0 &\leq -\frac{1}{2}(f\eta(x_0,t_0))^2 + (1+cS)f\eta(x_0,t_0) + cS^2 \\
0 &\leq -\frac{1}{4}(f\eta(x_0,t_0))^2 + \left[cS^2 + (1+cS)^2\right]
\end{aligned}$$

That is $f\eta(x,t) \leq f\eta(x_0,t_0) \leq C(n,h,S)$. Since $\eta = 0$ on $\partial B(x_0, 2r)$ and $f(x,0) = 0$. This implies that $\sup_{B(x_0,r)} f(x,t) \leq C(n,h,S)$. Using $1+\epsilon(n)$ fairness and the definition of f, we obtain the result. □

We now further obtain interior estimates of higher derivatives of $g(t)$.

Lemma 2.4.4. *Let $g(t), t \in [0,S]$ be a $C^\infty(D) \times [0,S]$ solution to the h flow, for some h which is $1+\epsilon(n)$ fair to $g(t)$, for all $t \in [0,S]$, $\epsilon(n)$ as in lemma 2.4.1. Then*

$$\sup_{B(x_0,r)} |\overline{\nabla}^i g|^2 \leq c(n,i,r,k_0,k_1,...k_i)\frac{1}{t^p} \quad ,\forall t \in (0,S], i \in \mathbb{N},$$

where $p = p(i,n)$ is an integer and $B(x_0,r)$ denotes a ball of radius r with respect to metric h contained in D. The norm is calculated using metric h. The constant $c(n,i,r,k_0,k_1,...k_i)$ decrease with the radius r.

Proof. Without loss of generality, we can assume $S \leq 1$. We calculate similar to [32]. By the result of Lemma (2.2.2), $\forall (x,t) \in M \times [0,S]$

$$\begin{aligned}
\frac{\partial}{\partial t}|\overline{\nabla}^m g|^2 \leq{}& g^{\alpha\beta}\overline{\nabla}_\alpha \overline{\nabla}_\beta |\overline{\nabla}^m g|^2 - 2g^{\alpha\beta}\overline{\nabla}_\alpha(\overline{\nabla}^m g)\overline{\nabla}_\beta(\overline{\nabla}^m g) \\
&+ c(m,n,h) \sum_{i+j\leq m, i,j\leq m} |\overline{\nabla}^i g||\overline{\nabla}^j g||\overline{\nabla}^m g| \\
&+ c(m,n,h) \sum_{i+j+k+l=m+2, i,j,k,l\leq m+1} |\overline{\nabla}^i g||\overline{\nabla}^j g||\overline{\nabla}^k g||\overline{\nabla}^l g||\overline{\nabla}^m g|.
\end{aligned}$$

We will prove the interior estimate by induction on m. Let $\Omega = B(x_0, 2r)$. Assume that we already have

$$|\overline{\nabla}^i g|^2 \leq \frac{c(n,m,h)}{t^{p(i,n)}} \quad ,\forall x \in \Omega,\ t \in [0,S], i = 1,2,...m-1$$

For simplicity, we denote $\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta$ by $\Box$. By assumption, we have

$$\Box|\overline{\nabla}^m g|^2 \leq -2g^{ij}\overline{\nabla}_i(\overline{\nabla}^m g)\overline{\nabla}_j(\overline{\nabla}^m g) + \frac{c}{t^q}|\overline{\nabla}^m g| + \frac{c}{t^q}|\overline{\nabla}^m g|^2 + \frac{c}{t}|\overline{\nabla}^m g||\overline{\nabla}^{m+1} g|$$

Where $q = q(n,m,h) \in \mathbb{N}$ denotes some power of p. In what follows we shall freely replace powers of q by q and powers of c by c. Using $1+\epsilon(n)$ fair condition, we can have

$$g^{ij}\overline{\nabla}_i(\overline{\nabla}^m g)\overline{\nabla}_j(\overline{\nabla}^m g) \geq \frac{1}{1+\epsilon}|\overline{\nabla}^{m+1} g|^2$$

It implies that

$$\Box|\overline{\nabla}^m g|^2 \leq -\frac{1}{2(1+\epsilon)}|\overline{\nabla}^{m+1} g|^2 + \frac{c}{t^q}|\overline{\nabla}^m g|^2 + \frac{c}{t^q}, \quad \forall (x,t) \in \Omega \times [0,S]$$

Similarly,

$$\Box|\overline{\nabla}^{m-1} g|^2 \leq -\frac{1}{2(1+\epsilon)}|\overline{\nabla}^m g|^2 + \frac{c}{t^q}, \quad \forall (x,t) \in \Omega \times [0,S]$$

in view of induction hypothesis. Following Shi in [32], we define

$$\psi(x,t) = (a + |\overline{\nabla}^{m-1} g|^2)|\overline{\nabla}^m g|^2,$$

where a is a constant to be chosen later. Combining two evolution equations , we get

$$\begin{aligned}
\Box\psi &= (a + |\overline{\nabla}^{m-1} g|^2)\Box|\overline{\nabla}^m g|^2 + |\overline{\nabla}^m g|^2\Box|\overline{\nabla}^{m-1} g|^2 - 2g^{ij}\overline{\nabla}_i|\overline{\nabla}^{m-1} g|^2\overline{\nabla}_j|\overline{\nabla}_j^m g|^2 \\
&\leq (a + |\overline{\nabla}^{m-1} g|^2)\left[-\frac{1}{2(1+\epsilon)}|\overline{\nabla}^{m+1} g|^2 + \frac{c}{t^q}|\overline{\nabla}^m g|^2 + \frac{c}{t^q}\right] \\
&\quad + |\overline{\nabla}^m g|^2\left[-\frac{1}{2(1+\epsilon)}|\overline{\nabla}^m g|^2 + \frac{c}{t^q}\right] + 4(1+\epsilon)|\overline{\nabla}^m g|^2|\overline{\nabla}^{m+1} g||\overline{\nabla}^{m-1} g|.
\end{aligned}$$

By cauchy inequality, the last term satisfies

$$4(1+\epsilon)|\overline{\nabla}^m g|^2|\overline{\nabla}^{m+1} g||\overline{\nabla}^{m-1} g| \leq \frac{1}{8(1+\epsilon)}|\overline{\nabla}^m g|^4 + \frac{c}{t^q}|\overline{\nabla}^{m+1} g|.$$

So,

$$\begin{aligned}
\Box\psi \leq &\left[\frac{c}{t^q} - \frac{a}{2(1+\epsilon)}\right]|\overline{\nabla}^{m+1} g|^2 \\
&- \frac{3}{8(1+\epsilon)}|\overline{\nabla}^m g|^4 + \frac{c}{t^q}(a + \frac{c}{t^q})|\overline{\nabla}^m g|^2 + (a + \frac{c}{t^q})\frac{c}{t^q}.
\end{aligned}$$

Using Cauchy Schwartz inequality and Young's inequality, we deduce that

$$\Box\psi \le \left[\frac{c}{t^q} - \frac{a}{2(1+\epsilon)}\right]|\overline{\nabla}^{m+1}g|^2 - \frac{1}{4(1+\epsilon)}|\overline{\nabla}^m g|^4 + \frac{c(1+a^4)}{t^q}, \tag{2.17}$$

for some $q = q(n,m,h) \in \mathbb{N}$.

Now we modify the function little bit which sets a to be a function of t.

Define $w(x,t) = t^{q+1}\left[\frac{2c(1+\epsilon)}{t^q} + |\overline{\nabla}^{m-1}g|^2\right]|\overline{\nabla}^m g|^2$, where c, q are constant given in (2.17).

Noted that $a(t) = \dfrac{2c(1+\epsilon)}{t^q} \ge |\overline{\nabla}^i g|^2$, for $i = 1, 2, ...m-1$.

The evolution equation of the function $\psi(x,t) = [a(t)+|\overline{\nabla}^{m-1}g|^2]|\overline{\nabla}^m g|^2$ becomes

$$\begin{aligned}
\Box\psi &= [a(t) + |\overline{\nabla}^{m-1}g|^2]\frac{\partial}{\partial t}|\overline{\nabla}^m g|^2 + [\frac{\partial}{\partial t}(a + |\overline{\nabla}^{m-1}g|^2)]|\overline{\nabla}^m g|^2 - g^{ij}\overline{\nabla}_i\overline{\nabla}_j\psi \\
&\le |\overline{\nabla}^m g|^2\frac{\partial}{\partial t}a(t) + \left[\frac{c}{t^q} - \frac{a}{2(1+\epsilon)}\right]|\overline{\nabla}^{m+1}g|^2 - \frac{1}{4(1+\epsilon)}|\overline{\nabla}^m g|^4 + \frac{c(1+a^4)}{t^q} \\
&\le -\frac{1}{4(1+\epsilon)}|\overline{\nabla}^m g|^4 + \frac{c(1+a^4)}{t^q} \\
&\le -\frac{1}{4(1+\epsilon)}|\overline{\nabla}^m g|^4 + \frac{c}{t^{5q}}
\end{aligned}$$

Then we can evaluate the evolution equation of $w(x,t)$.

$$\begin{aligned}
\Box w(x,t) &\le (q+1)t^q\psi + \frac{c}{t^{4q-1}} - \frac{t^{q+1}}{4(1+\epsilon)}|\overline{\nabla}^m g|^4 \\
&= (q+1)\frac{w}{t} + \frac{c}{t^{4q}} - \frac{1}{4(1+\epsilon)}\frac{w^2}{t^{q+1}[a(t) + |\overline{\nabla}^{m-1}g|^2]^2} \\
&\le (q+1)\frac{w}{t} + \frac{c}{t^{4q}} - \frac{w^2}{c^2}t^{q-1} \qquad ,\forall(x,t) \in \Omega\times[0,S].
\end{aligned}$$

Let $f(x,t) = t^{5q}w(x,t)$, and calculate its evolution equation.

$$\Box f \le \frac{q+1}{t}f + ct^q - c\frac{f^2}{t^{4q+1}} + \frac{5qf}{t}$$

Let η be the cut-off function in Lemma (2.4.3). Let $\Phi = f\eta$, and calculate the evolution equation of $\Phi(x,t)$.

$$\begin{aligned}\Box\Phi &= \Box f\cdot\eta + f\cdot\Box\eta - 2g^{ij}\overline{\nabla}_i\eta\overline{\nabla}_j f\\ &\leq -fg^{ij}\overline{\nabla}_i\overline{\nabla}_j\eta + \frac{(q+1)f\eta}{t} + ct^q\eta - \frac{cf^2\eta}{t^{4q+1}} + \frac{5qf\eta}{t}\\ &\quad + \frac{2(1+\epsilon)}{\eta}|\overline{\nabla}\eta|^2 f - \frac{2}{\eta}g^{ij}\overline{\nabla}_i\eta\overline{\nabla}_j\Phi\end{aligned}$$

Let (x_0,t_0) be the interior point of $B(x_0,2r)\times[0,S]$ where Φ attains its maximum. By the properties of η, we conclude that

$$0\leq Cf + C\frac{\Phi(x_0,t_0)}{t_0} + C - \frac{c'\Phi(x_0,t_0)^2}{\eta(x_0)t_0^{4q+1}} + C\frac{\Phi(x_0,t_0)}{t_0}$$

where $C=C(n,h,m,r)$, $c'=c(n,m,h)>0$.

Using the fact that $S\leq 1$ and $\eta\in[0,1]$,we get

$$\Phi(x_0,t_0)\leq C'' = C''(n,h,m,r).$$

It implies that

$$f(x,t)\leq C'',\quad \forall (x,t)\in B(x_0,r)\times[0,S]$$

The result follows from the definition of $f(x,t)$. $\Box$

Theorem 2.4.5. *Let $g(t), t\in[0,S]$, h be as in lemma 2.4.4. Then*

$$\sup_{x\in M}|\overline{\nabla}^i g(x,t)|^2 \leq \frac{c(n,i,k_0,k_1,...k_i)}{t^i}\quad ,\forall t\in(0,S], i\in\mathbb{N}.$$

Proof. Without loss of generality, we assume that $S\leq 1$. For any given $t_0\in[0,S]$, let $R=t_0\leq 1$. Let $\tilde{h}=\frac{1}{R}h$, $\tilde{g}(t)=\frac{1}{R}g(Rt)$. Then $\tilde{h}$ is $(1+\epsilon)$-fair to $\tilde{g}(t)$ and $\tilde{g}(t)$ solves $\tilde{h}$ flow. Noted that

$$\tilde{k_i} = \sup_{x\in M}|^{\tilde{h}}\nabla^i Riem(\tilde{h})|^2_{\tilde{h}} \leq k_i.$$

Hence by lemma 2.4.4, we get

$$|^{\tilde{h}}\nabla^i\tilde{g}|^2_{\tilde{h}}(x,1)\leq c(n,i,\tilde{k}_0,...\tilde{k}_i)\leq c(n,i,k_0,k_1,...k_i).$$

But

$$|^{\tilde{h}}\nabla^i\tilde{g}|^2_{\tilde{h}}(x,1) = R^i|\overline{\nabla}^i g|^2_h(x,R) \leq c(n,i,k_0,...k_i)$$

So ,

$$|\overline{\nabla}^i g(x,t)|^2 \leq \frac{c(n,i,k_0,k_1,...k_i)}{t^i} \quad \forall (x,t) \in M \times (0,S].$$

□

2.5 Solution to Dirichlet problem.

As soon as we established the priori estimates, we are able to prove the following existence theorem.

Theorem 2.5.1. *Let g_0 be a $C^\infty(D)$ metric and h a metric which is $1+\epsilon(n)$ fair to g_0 on D, where $D \subset M$ is a compact domain in M ($\epsilon(n)$ as in lemma 2.4.1 and smaller than $\tilde{\delta}(n)$ in Lemma 2.3.2). There exists an $S = S(n,k_0) > 0$ and a family of metrics $g(t), t \in [0,S]$ which solves h flow, h is $1+3\epsilon(n)$ fair to $g(t)$ for all $t \in [0,S]$, and $g|_{\partial D}(t) = g_0$, $g(0) = g_0$.*

Proof. Let $S = S(n,k_0,\epsilon)$ be the positive real number obtained in Theorem 2.3.3.

$$\begin{aligned}\mathcal{U}_S =&\{u : D \times [0,S] \to \otimes^2 T^*(D)| \ \frac{1}{2}h_{ij} \leq u_{ij} \leq 2h_{ij}, u(x,0) = g_0, x \in D,\\ &u(x,t) = g_0, x \in \partial D, ||u|| < +\infty\}\end{aligned}$$

$$\mathcal{B} = \{u : D \times [0,S] \to \otimes^2 T^*(D)| \ u(x,0) = g_0, x \in D, u(x,t) = g_0, x \in \partial D, ||u|| < +\infty\}$$

where $||u|| = \sup_{D\times[0,S]} |u|_h + \sup_{D\times[0,S]} |\overline{\nabla} u|_h + \sup_{D\times[0,S]} |\overline{\nabla}^2 u|_h$.

Let $\Phi : \mathcal{U}_S \times [0,1] \to \mathcal{B}$ be a solution operator such that $v = \Phi(u,s)$ satisfies the following parabolic system.

$$\begin{cases} \dfrac{\partial}{\partial t}v_{kl} &= \hat{u}^{ij}\overline{\nabla}_i\overline{\nabla}_j v_{kl} - s\hat{u}^{cd}\hat{u}_{kp}h^{pq}\overline{R}_{lcqd} - s\hat{u}^{cd}\hat{u}_{lp}h^{pq}\overline{R}_{kcqd} \\ & +\frac{s}{2}\hat{u}^{cd}\hat{u}^{pq}(\overline{\nabla}_k u_{pc}\cdot\overline{\nabla}_l u_{qd} + 2\overline{\nabla}_c u_{kp}\cdot\overline{\nabla}_q u_{ld} \\ & -2\overline{\nabla}_c u_{kp}\cdot\overline{\nabla}_d u_{lq} - 4\overline{\nabla}_k u_{pc}\cdot\overline{\nabla}_d u_{lq}) \\ v(x,0) &= sg_0 + (1-s)h(x),\ x\in D \\ v(x,t) &= sg_0 + (1-s)h(x),\ x\in\partial D \end{cases} \tag{2.18}$$

where $\hat{u}(x,t) = (1-s)h(x) + su(x,t),\ \ s\in[0,1]$. $v = \Phi(u,s)$ is well defined for all $u\in\mathcal{U}_S, s\in[0,1]$ (see Theorem 7.1 in [18]) and $\Phi(u,0) = h,\ \forall u\in\mathcal{U}_S$. To apply Leray-Schauder fixed point Theorem, we need to check that Φ is a compact mapping. We first verify the continuity in u of $\Phi(u,s)$ on $\mathcal{U}_S\times[0,T]$. Let $v_1 = \Phi(u_1,s),\ v_2 = \Phi(u_2,s)$, for $v = v_1 - v_2$, $u = u_1 - u_2$, using the fact that $u^{ij} - v^{ij} = u^{i\alpha}v^{j\beta}(u_{\alpha\beta} - v_{\alpha\beta})$, we have

$$\begin{aligned} \frac{\partial}{\partial t}v_{kl} - \hat{u}_1^{ij}\overline{\nabla}_i\overline{\nabla}_j v_{kl} = {} & su * \hat{u}_1 * \hat{u}_1^{-1} * \hat{u}_2^{-1} * \overline{Rm} + su * \hat{u}_1^{-1} * \hat{u}_2^{-1} * \hat{u}_2 * \overline{Rm} \\ & + su * \hat{u}_1^{-1} * \hat{u}_2^{-1} * \overline{\nabla}^2 v_2 + su * \hat{u}_1^{-1} * \hat{u}_1^{-1} * \hat{u}_2^{-1} * \overline{\nabla}u_1 * \overline{\nabla}u_1 \\ & + su * \hat{u}_1^{-1} * \hat{u}_2^{-1} * \hat{u}_2^{-1} * \overline{\nabla}u_1 * \overline{\nabla}u_1 + \hat{u}_2^{-1} * \hat{u}_2^{-1} * \overline{\nabla}u * \overline{\nabla}u_1 \\ & + \hat{u}_2^{-1} * \hat{u}_2^{-1} * \overline{\nabla}u * \overline{\nabla}u_2 \quad . \end{aligned}$$

Also,

$$v(x,0) = 0, x\in D \quad \text{and} \quad v(x,t) = 0, x\in\partial D.$$

By the Schauder estimate in [17], we see that when u is small enough, v will be small. The uniform continuity in s of $\Phi(u,s)$ is proved analogously. The compactness follows from the apriori estimate for $\sup_{D\times[0,S]}||v||_{C^{2+\alpha}}$ (see [17]) and the Arzelà-Ascolii Theorem. It remains to establish the apriori estimate for the fixed point $u_s = \Phi(u_s,s)$. If $u_s = \Phi(u_s,s)$, one may verify that $g(x,t) =$

$su_s(x,t)+(1-s)h$ solves

$$\begin{cases} \dfrac{\partial}{\partial t}g_{kl} &= g^{ij}\overline{\nabla}_i\overline{\nabla}_j g_{kl} - s^2 g^{cd}g_{kp}h^{pq}\overline{R}_{lcqd} - s^2 g^{cd}g_{lp}h^{pq}\overline{R}_{kcqd} \\ & \quad +\frac{1}{2}g^{cd}g^{pq}(\overline{\nabla}_k g_{pc}\cdot\overline{\nabla}_l g_{qd} + 2\overline{\nabla}_c g_{kp}\cdot\overline{\nabla}_q g_{ld} \\ & \quad -2\overline{\nabla}_c g_{kp}\cdot\overline{\nabla}_d g_{lq} - 4\overline{\nabla}_k g_{pc}\cdot\overline{\nabla}_d g_{lq}) \\ g(x,0) &= s^2 g_0 + (1-s^2)h(x),\ x\in D \\ g(x,t) &= s^2 g_0 + (1-s^2)h(x),\ x\in \partial D \end{cases} \tag{2.19}$$

Using the same technique we used in Theorem 2.3.3, we can conclude that h is a priori $1+3s^2\epsilon$ fair to $g(x,t)$ for all $t\in[0,S]$. Thus $\sup\limits_{D\times[0,S]}|u_s|_h \le 1+3\epsilon$ for all $s\in[0,1]$. Also, argue as in Lemma 2.4.1 and Lemma 2.4.2, we can obtain $\sup\limits_{D\times[0,S]}|\overline{\nabla}u_s|_h \le C(n,h,\epsilon,g_0)$ and $\sup\limits_{D\times[0,S]}|\overline{\nabla}^2 u_s|_h \le C'(n,h,\epsilon,g_0)$ for all $s\in[0,1]$. Thus by using Leray Schauder fixed point theorem, there exists a fixed point $g=\Phi(g,1)$ which solves the h-flow with boundary data g_0.

□

2.6 Existence of entire solutions.

In this section, our final goal is to find a sensible solution to the h-flow with initial metric g_0 which is non-smooth. Before doing this, we first establish the existence of h-flow on M with smooth initial metric g_0.

Theorem 2.6.1. *Let g_0 be a $C^\infty(M)$ metric and h a metric on M which is $1+\epsilon(n)$ fair to g_0, $\epsilon(n)$ as in lemma 2.4.1. There exists $T=T(n,k_0)>0$ and a family of metrics $g(t), t\in[0,T]$ in $C^\infty(M\times[0,T])$ which solves h flow with initial metric $g(0)=g_0$, h is $(1+3\epsilon)$ fair to $g(t)$ for $t\in[0,T]$, and*

$$|\overline{\nabla}^i g|^2 \le \frac{c(n,i,k_0,...k_i)}{t^i} \quad ,\forall t\in(0,T], i\in\mathbb{N}.$$

Proof. If M is compact manifold, we obtain the result using Theorem 2.5.1 and Theorem 2.4.5. If not , let $\{D_i\}, i\in\mathbb{N}$ be a family of compact sets which exhaust

M, $D_i = B(h)(x_0, i)$ where $B(h)(x_0, i)$ is the ball of radius i for some fixed point x_0 with respect to the metric given by h.

Let $g_i(t),\ t \in [0, T]$ be the Dirichlet solutions on D_i with boundary data g_0. By theorem 2.5.1, T depends on n, k_0 only. Let D_j be a fixed compact set. For each $i >> j$, $g_i(t), t \in [0, T]$ solves h flow on D_j. On the other hand, since D_j is compact, we have

$$\sup_{D_j} |\overline{\nabla}^m g_0|^2 = C_{j,m} < +\infty$$

We now claim that $|\overline{\nabla}^m g_i(x,t)|^2 \le C'(m, n, h, j, C_{j,m})$ for all $(x, t) \in D_j \times [0, T]$, $i > 2j$. If we are able to show this, we may apply Arzelà-Ascolii Theorem to obtain a subsequence convergent to a smooth limit $g(t), t \in [0, T]$ on D_j. Apply this argument on each D_j, we can take a diagonal subsequence which converges to a limit solution $g(t), t \in [0, T]$ on M. So, it suffices to prove the claim.

We prove it by induction on m. Let $\psi(x, t) = [\phi(g_i) + a(n)]|\overline{\nabla} g_i(x, t)|^2$ as in Lemma 2.4.1 .

By equation (2.9), we have

$$(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha \overline{\nabla}_\beta)\psi \le -\frac{1}{2}\psi^2 + c_0(n, k_0, k_1) \quad , \forall (x, t) \in D_{2j} \times [0, T]$$

Let η be the cut-off function in Lemma 2.4.3 with $r = j$ now. Define $F(x, t) = \psi(x, t)\eta(x)$,

$$\begin{aligned} \Box F &\le -\frac{1}{2}\psi^2\eta + c_0\eta - 2g^{ij}\overline{\nabla}_i\eta\overline{\nabla}_j\psi - \psi g^{ij}\overline{\nabla}_i\overline{\nabla}_j\eta \\ &\le -\frac{1}{2\eta}F^2 + c_0\eta + C\psi - 2g^{ij}\frac{\overline{\nabla}_i\eta}{\eta}\overline{\nabla}_j F, \quad \forall (x, t) \in D_{2j} \times [0, T]. \end{aligned}$$

where $C = C(n, k_0, j) > 0$.

Assume (x_0, t_0) is an interior point of $B(h)(x_0, 2j) \times [0, T]$ where $F(x, t)$ attains its maximum. We get

$$F(x_0, t_0) \le C' = C'(n, k_0, k_1, j)$$

So, $\forall (x, t) \in D_j \times [0, T]$

$$\psi(x, t) \le \max(C', \sup_{D_j} \psi(x, 0)) = \tilde{C}(n, k_0, k_1, j, C_{j,1}).$$

This implies

$$|\overline{\nabla} g_i(x,t)|^2 \le C'(n,h,j,C_{j,1}) \quad ,\forall (x,t) \in D_j \times [0,T]$$

Assume we already have

$$\sup_{D_{2j}} |\overline{\nabla}^k g_i(x,t)|^2 \le \hat{C}(m,n,h,j,g_0) \quad \forall (x,t) \in D_{2j} \times [0,T], \forall k \in \{1,2,...m-1\}.$$

As in the proof of lemma 2.4.4, we define $\phi(x,t) = (a + |\overline{\nabla}^{m-1} g|^2)|\overline{\nabla}^m g|^2$ where a is a constant to be chosen. By (2.17), we have

$$\Box\phi \le \left[\frac{c}{t^q} - \frac{a}{2(1+\epsilon)}\right] |\overline{\nabla}^{m+1} g|^2 - \frac{1}{4(1+\epsilon)} |\overline{\nabla}^m g|^4 + \frac{c(1+a^4)}{t^q},$$

Since we have upper bound independent of time, the constant q in the above equation is indeed 0, and c is in fact some power of $\hat{C}$. We may assume that $c > \hat{C}$.

$$\Box\phi \le \left[c - \frac{a}{2(1+\epsilon)}\right] |\overline{\nabla}^{m+1} g|^2 - \frac{1}{4(1+\epsilon)} |\overline{\nabla}^m g|^4 + c(1+a^4),$$

Choose $a = 2c(1+\epsilon)$. Noted that $a > \hat{C}$.

$$\begin{aligned}\Box\phi &\le -\frac{1}{4(1+\epsilon)} |\overline{\nabla}^m g|^4 + c'(m,j,n,h,g_0) \\ &\le -\frac{\phi^2}{16(1+\epsilon)a^2} + c' \\ &= -C(m,j,n,h,g_0)\phi^2 + c', \quad \forall (x,t) \in D_{2j} \times [0,T]\end{aligned}$$

Similarly we define $g = \phi\eta$. And follow exactly the same step, we may conclude that

$$\phi(x,t) \le \tilde{C}(m,n,h,j,C_{j,m}), \quad \forall (x,t) \in D_j \times [0,T].$$

The claim follows immediately. $\Box$

Theorem 2.6.2. *Let g_0 be a complete continuous metric and h a complete metric on M which is $1+\epsilon(n)$ fair to g_0 , $\epsilon(n)$ as in lemma 2.1.4. There exists $T =$*

$T(n,k_0) > 0$ and a family of metrics $g(t), t \in (0,T]$ in $C^\infty(M \times (0,T])$ which solves h flow for $t \in (0,T]$,h is $(1+4\epsilon)$ fair to $g(t)$ for $t \in (0,T]$, and

$$\lim_{t\to 0} \sup_{x\in\Omega'} |g(t) - g_0| = 0,$$
$$\sup_{x\in M} |\overline{\nabla}^i g|^2 \le \frac{c(n,i,k_0,...k_i)}{t^i} \quad ,\forall t \in (0,T], i \in \mathbb{N},$$

where Ω' is any open set satisfying $\Omega' \subset \Omega$, where Ω is any open set in which g_0 is continuous.

Proof. Let p be a fixed point on M, $\{D_a = B_h(p,a)\}$ be a family of compact sets which exhaust M. Let $\varphi_a : M \to \mathbb{R}$ be a smooth cut-off function on M, such that $\varphi_a = 0$ outside $B_h(p,a)$ and $\varphi_a = 1$ on each $B_h(p,a/2)$. Since D_a is compact, there exists $\delta_a > 0$ such that $inj(p) \ge \delta_a > 0$ for all $p \in D_a$. Thus, we can define a sequence of smooth metrics which approximating g_0 by

$$^a g_0(p) = \frac{\varphi_a(p)}{\epsilon_a^n} \int_{q\in M} g_0(q)\phi\left(\frac{|exp_p^{-1}(q)|}{\epsilon_a}\right) dq + (1-\varphi_a)h \quad ,\forall p \in M$$

where $exp_p : T_pM \to M$ is the exponential map of M at p, ϵ_a is chosen small enough such that $\epsilon_a < \delta_a$ and converges to 0 as a goes to infinity. ϕ is a nonnegative smooth function on $\mathbb{R}^n$ with support on unit ball $B_h(0)$ satisfying $\int_{\mathbb{R}^n} \phi = 1$. $\{^a g_0\}_{a\in\mathbb{N}}$ is a sequence of smooth metrics which satisfy $\lim_{a\to\infty} {}^a g_0 = g_0$, where the limit is uniform in the C^0 norm on any compact set. It follows that h is $(1+\frac{4\epsilon}{3})$ fair to $^a g_0$ for all $a \ge N$ for some $N \in \mathbb{N}$. We flow each $^a g_0$ by h-flow to obtain a family of metrics $^a g(t),\ t \in [0,T],\ T = T(n,k_0)$ independent of a which satisfy

$$|\overline{\nabla}^j(^a g(t))|^2 \le \frac{c_j}{t^j}, \quad \forall t \in (0,T],$$

independent of a, for all $a \ge N$ and h is $1+4\epsilon$ fair to each $^a g(t)$. We then obtain a limiting solution $g(x,t), t \in (0,T]$ via $g(x,t) = \lim_{a\to\infty} {}^a g(t)$, which is defined for all $t \in (0,T]$. This limit is obtained by using Arzelà-Ascolii Theorem and it maybe necessary to pass to sub-sequence to obtain the limit.

It remains to show that the metrics $g(t)|_{\Omega'}$ uniformly approaches $g_0|_{\Omega'}$ as $t \to 0$. We first obtain estimates on the rate at which it approaches to the limit if g_0 is smooth.

Let $\epsilon > 0$ be given as in previous lemmas. Using the evolution equation of h-flow and $1+\epsilon$ fairness, g^{ij} satisfies

$$\frac{\partial}{\partial t} g^{ij} \le g^{\alpha\beta}\overline{\nabla}_\alpha \overline{\nabla}_\beta g^{ij} + c(n,h) g^{ij} - S^{ij}, \tag{2.20}$$

where S^{ij} is a positive tensor. Fix x_0 in Ω', and fix a coordinate chart around x_0, $\psi : U \to M, x_0 \in U \subset\subset \Omega'$. Define a $(0,2)$ tensor l by

$$l(V,W)(x) = v_j(x)W_j(x)h_{qp}(x_0)g_0^{qi}(x_0)h^{pj}(x),$$

Noted that the tensor l depends on the coordinate chart and $l^{ij}(x_0) = g_0^{ij}(x_0)$. By definition of l, we get

$${}^h|g_0^{ij}(x) - l^{ij}(x)| \le {}^h|g_0^{ij}(x) - g_0^{ij}(x_0)| +^h |g_0^{ij}(x_0) - h_{qp}(x_0)g_0^{qi}(x_0)h^{pj}(x)| \le \frac{\epsilon}{2}. \tag{2.21}$$

for all $x \in B(h)(x_0, r) \subset U$ for some small $r = r(g_0, h, \epsilon) > 0$, where the last inequality follows from the continuity of g_0^{ij} and continuity of h^{ij}. This implies

$$(1-2\epsilon)h \le l \le (1+2\epsilon)h \quad , \forall x \in B(h)(x_0, r)$$

And as a consequence of definition of l and U being compact, we also have

$$\sup_{B(h)(x_0,r)}^{h} |\overline{\nabla}\, \overline{\nabla} l| \le c(h,n,U).$$

By (2.20), we get

$$\frac{\partial}{\partial t}(g^{ij} - l^{ij}) \le g^{\alpha\beta}\overline{\nabla}_\alpha \overline{\nabla}_\beta (g^{ij} - l^{ij}) + c(h,n,U)(g^{ij} - l^{ij}) + c(h,n,U)h^{ij},$$

and hence

$$\frac{\partial}{\partial t}(e^{-ct}(g^{ij} - l^{ij}) - cth^{ij}) \le g^{\alpha}\overline{\nabla}_\alpha \overline{\nabla}_\beta (e^{-ct}(g^{ij} - l^{ij}) - cth^{ij}),$$

for all $x \in B(h)(x_0, r)$. Define (0,2) tensor f by $f^{ij} = e^{-ct}(g^{ij} - l^{ij}) - cth^{ij}$,we have $(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)f^{ij} \leq 0$.

Let η be the cut-off function as previous lemma for the ball$B(h)(x_0, r)$, with $\eta \equiv 1$ on $B(h)(x_0, r/2)$ and $\eta \equiv 0$ on $\partial B(h)(x_0, r)$. Using the properties of η, we see that

$$\begin{aligned}(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)(\eta f^{ij}) &\leq -f^{ij}g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\eta - 2g^{\alpha\beta}\overline{\nabla}_\alpha\eta\overline{\nabla}_\beta f^{ij} \\ &\leq cf^{ij} - \frac{2}{\eta}g^{\alpha\beta}\overline{\nabla}_\alpha\eta\overline{\nabla}_\beta(\eta f^{ij}), \quad \text{where } c = c(n, h, \frac{1}{r}, U).\end{aligned}$$

It implies that

$$(\frac{\partial}{\partial t} - g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta)\eta(f^{ij} - c_1th^{ij}) \leq 0, \quad \text{for some } c_1 = c_1(\frac{1}{r}, n, h, U).$$

Hence, by maximum principle, we get

$$\eta f^{ij}(x,t) - c_1th^{ij}(x) \leq \eta f^{ij}(x,0) \leq \frac{\epsilon}{2}h^{ij}(x), \quad \forall x \in B(h)(x_0, r).$$

So,

$$\begin{aligned}&f^{ij}(x,t) \leq (c_1t + \frac{\epsilon}{2})h^{ij}, \quad \forall x \in B(h)(x_0, \frac{r}{2}), t \in [0, T] \\ &f^{ij}(x,t) \leq \epsilon h^{ij} \quad , \quad \forall x \in B(h)(x_0, \frac{r}{2}), t \leq \frac{\epsilon}{2c_1}, \\ &g^{ij} - l^{ij} \leq e^{ct}(\epsilon + ct)h^{ij} \leq 2\epsilon h^{ij}, \qquad \forall x \in B(h)(x_0, \frac{r}{2}), t \leq T(c_1, c, \epsilon).\end{aligned}$$

Substitute (2.21) into the above inequality, we get that

$$g^{ij} - g_0^{ij} = g^{ij} - l^{ij} + l^{ij} - g_0^{ij} \leq 3\epsilon h^{ij} \quad , \forall x \in B(h)(x_0, \frac{r}{2}), \quad t \leq T(c_1, c, \epsilon). \tag{2.22}$$

Apply the above argument to each ${}^a g(t)$. We have

$${}^a g^{ij} \leq {}^a g_0^{ij} + 3\epsilon h^{ij} \quad , \forall x \in B(h)(x_0, \frac{r_a}{2}), \quad t \leq T(n, U, h, \epsilon, \frac{1}{r_a}).$$

where r_a is chosen such that

$${}^h|{}^a g_0^{ij} - {}^a l^{ij}| \leq \frac{\epsilon}{2} \quad , \forall x \in B(h)(x_0, r_a)$$

We wish to choose r so that it is independent of a. $\forall x \in B(h)(x_0, r_a)$,$b > a$

$$\begin{aligned} {}^h|{}^b g_0^{ij}(x) -^b l^{ij}| &\leq^h |{}^b g_0^{ij} -^a g_0^{ij}| +^h |{}^a g_0^{ij} -^a l^{ij}| +^h |{}^b l^{ij} -^a l^{ij}| \\ &\leq 3\epsilon \qquad \text{(provided that } a, b \text{ are large enough).} \end{aligned}$$

So we can choose $r > 0$ such that it is independent of a.

Hence, $g(t) = \lim_{a\to\infty} {}^a g(t)$ satisfies

$$g^{ij} - g_0^{ij} \leq 3\epsilon h^{ij} \quad ,\forall x \in B(h)\left(x_0, \frac{r}{2}\right), \quad t \leq T(c_1, c, \epsilon). \tag{2.23}$$

Let ϕ be the function defined in lemma 2.4.1. By calculation in the lemma , we see that ϕ satisfies

$$\frac{\partial}{\partial t}\phi \leq g^{\alpha\beta}\overline{\nabla}_\alpha\overline{\nabla}_\beta\phi + c_0(h, n) - \frac{m^2}{8}|\overline{\nabla} g|^2,$$

Arguing as above, but for ϕ instead of g^{ij}, we get

$$\phi(x, t) \leq \phi(x, 0) + 3\epsilon, \quad \text{for some } S = S(n, h, g_0, \epsilon, \Omega') > 0. \tag{2.24}$$

Combining (2.24) and (2.23), we see that

$$\sup_{\Omega'} {}^h|g_0(x) - g(x, t)| \leq c(n)\epsilon^{\frac{1}{m(n)}}, \quad \forall t \in [0, S], \forall x \in \Omega'$$

$\square$

Chapter 3

Existence of Ricci Flow in the case of $K_g^{\mathbb{C}} > 0$

In this chapter, we will study the short time existence problem of Ricci flow on open manifolds of positive complex sectional curvature without requiring the upper curvature bound. The idea of proof is to consider the graph of a convex function β on C_i where its doubling is a smooth closed manifold (M_i, g_i) with $K_{g_i}^{\mathbb{C}} > 0$ converging to (M, g). And then estimate the lower bound for the lifespan of Ricci flow on each (M_i, g_i) to ensure the maximal time will not degenerate to 0 when we let $i \to \infty$. After that, we obtain curvature bound independent of i of arbitrarily large ball around the soul point p_0 which allows us to obtain a limit solution on M.

For the case of $K^{\mathbb{C}} \geq 0$, several additional difficulties arise. For instance, the soul is not necessarily a point. But it can be overcame via splitting theorem (see Theorem 5.1 in [10]). Another difficulty is that the sublevels set of Busemann function $C_l = b^{-1}((-\infty, l])$ have non-smooth boundary. By reparameterizing the distance function $d(\,\cdot\,, C_l)$, an sequence of C^∞ closed manifolds $D_{l,k}$ can be constructed which converges to the double $D(C_l)$. But $D_{l,k}$ are no longer convex. Fortunately, its complex sectional curvature can be controlled by estimating the hessian of $d^2(\,\cdot\,,C_l)$. And, a curvature control can be obtained for some time for the Ricci flows on $(D_{l,k},g_{l,k})$. As a consequence, we can obtain a limit Ricci flow $(D_{l,\infty},g_{l,\infty})$ with $K^{\mathbb{C}}(g_{l,\infty}(t))>0$ for $t>0$. It reduces the case of $K^{\mathbb{C}}\geq 0$ to the positively curved case.

3.1 Basic Background material

We first need to introduce the definition of complex sectional curvature on a manifold.

Definition 3.1.1. *Let (M^n,g) be a Riemannian manifold and consider its complexified tangent bundle $T^{\mathbb{C}}M=TM\otimes\mathbb{C}$. We extend the curvature tensor R and the metric g at p to $\mathbb{C}$-multilinear maps $R:(T_p^{\mathbb{C}}M)^4\to\mathbb{C}, g:(T_p^{\mathbb{C}}M)^2\to\mathbb{C}$. The complex sectional curvature of a 2-dimensional complex subspace σ of $T_p^{\mathbb{C}}M$ is defined by*

$$K^{\mathbb{C}}(\sigma)=R(u,v,\bar{u},\bar{v}),$$

where u and v form any unitary basis for σ. We say M has non-negative complex sectional curvature if $K^{\mathbb{C}}\geq 0$.

Definition 3.1.2. *Let (M^n,g) be a Riemannian manifold with $n\geq 4$. We say M has nonnegative isotropic curvature if*

$$\begin{aligned} R(e_1,e_3,e_1,e_3)+R(e_1,e_4,e_1,e_4)+R(e_2,e_3,e_2,e_3)&\\ +R(e_2,e_4,e_2,e_4)-2R(e_1,e_2,e_1,e_4)&\geq 0\end{aligned}$$

for all orthonormal four-frames $\{e_1,e_2,e_3,e_4\}$ on M.

For manifold M^n with $n\geq 4$, the curvature on $M\times\mathbb{R}^2$ is given by

$$\tilde{R}(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_4)=R(v_1,v_2,v_3,v_4) \tag{3.1}$$

where $\tilde{v}_i=(v_i,e_i)\in T_{(p,q)}(M\times\mathbb{R}^2)=T_pM\times T_q\mathbb{R}^2$.

The following proposition gives equivalence of nonnegative isotropic curvature and nonnegative complex sectional curvature.

Proposition 3.1.3. *(see Proposition 7.18 in [29]) Let M^n be a Riemannian manifold with $n \geq 4$, $\tilde{R}$ be the curvature tensor on $M \times \mathbb{R}^2$ defined by (3.1). Then the followings are equivalent.*

1. *$\tilde{R}$ has nonnegative isotropic curvature.*

2. *We have*

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) + \mu^2 R(e_2, e_3, e_2, e_3) \\ +\lambda^2\mu^2 R(e_2, e_4, e_2, e_4) - 2\mu\lambda R(e_1, e_2, e_3, e_4) \geq 0$$

   for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ on M, for all $\mu, \lambda \in [-1, 1]$.

3. *We have $R(\eta, \zeta, \bar{\eta}, \bar{\zeta}) \geq 0$ for all $\eta, \zeta \in T_p^{\mathbb{C}}M$,$p \in M$.*

Proof. (1) $\Rightarrow$ (2): Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal four-frames on M, $\mu, \lambda \in [-1, 1]$. We define

$$\tilde{e}_1 = \big(e_1, (0, 0)\big), \quad \tilde{e}_2 = \big(\mu e_2, (0, \sqrt{1-\mu^2})\big), \\ \tilde{e}_3 = \big(e_3, (0, 0)\big), \quad \tilde{e}_4 = \big(\lambda e_4, (0, \sqrt{1-\lambda^2})\big).$$

The vectors $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ form an orthonormal four-frame in $M \times \mathbb{R}^2$. By (1), we have

$$\tilde{R}(\tilde{e}_1, \tilde{e}_3, \tilde{e}_1, \tilde{e}_3) + \tilde{R}(\tilde{e}_1, \tilde{e}_4, \tilde{e}_1, \tilde{e}_4) + \tilde{R}(\tilde{e}_2, \tilde{e}_3, \tilde{e}_2, \tilde{e}_3) \\ +\tilde{R}(\tilde{e}_2, \tilde{e}_4, \tilde{e}_2, \tilde{e}_4) - 2\tilde{R}(\tilde{e}_1, \tilde{e}_2, \tilde{e}_1, \tilde{e}_4) \geq 0$$

which implies

$$R(e_1, e_3, e_1, e_3) + \lambda^2 R(e_1, e_4, e_1, e_4) + \mu^2 R(e_2, e_3, e_2, e_3) \\ +\lambda^2\mu^2 R(e_2, e_4, e_2, e_4) - 2\mu\lambda R(e_1, e_2, e_3, e_4) \geq 0.$$

(2) $\Rightarrow$ (3): Let $\eta, \zeta \in T_p^{\mathbb{C}}M$ be 2 linearly independent vectors. Let $\sigma \subset T_p^{\mathbb{C}}M$ be the plane spanned by η, ζ. By proposition B.3 in [29], there exists an orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and real numbers $\lambda, \mu \in [0,1]$ such that $e_1 + i\mu e_2$ and $e_3 + i\lambda e_4$ are in σ. Let $z = e_1 + i\mu e_2$ and $w = e_3 + i\lambda e_4$, we can find $a, b, c, d \in \mathbb{C}$ such that $\zeta = az + bw$ and $\eta = cz + dw$. This implies

$$R(\eta, \zeta, \bar{\eta}, \bar{\zeta}) = |ad - bc|^2 R(z, w, \bar{z}, \bar{w}).$$

Using Bianchi identity, we can obtain

$$\begin{aligned} R(z, w, \bar{z}, \bar{w}) = R(e_1, e_3, e_1, e_3) &+ \lambda^2 R(e_1, e_4, e_1, e_4) \\ &+ \mu^2 R(e_2, e_3, e_2, e_3) + \mu^2\lambda^2 R(e_2, e_4, e_2, e_4) \\ &- 2\mu\lambda R(e_1, e_2, e_3, e_4) \geq 0 \end{aligned}$$

which implies (3).

(3) $\Rightarrow$ (1): Let $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ be an orthonormal four-frames on $M \times \mathbb{R}^2$. We write $\tilde{e}_i = (x_i, y_i) \in T_{(p,q)}M \times \mathbb{R}^2$ where $x_i \in T_pM$, $y_i \in T_q\mathbb{R}^2$. Define $\zeta = x_1 + ix_2$ and $\eta = x_3 + ix_4$. It follows from the first Bianchi identity that

$$\begin{aligned} 0 \leq R(\eta, \zeta, \bar{\eta}, \bar{\zeta}) = &R(x_1, x_3, x_1, x_3) + R(x_1, x_4, x_1, x_4) \\ &+ R(x_2, x_3, x_2, x_3) + R(x_2, x_4, x_2, x_4) \\ &- 2R(x_1, x_2, x_3, x_4). \end{aligned}$$

It implies

$$\begin{aligned} \tilde{R}(\tilde{e}_1, \tilde{e}_3, \tilde{e}_1, \tilde{e}_3) + \tilde{R}(\tilde{e}_1, \tilde{e}_4, \tilde{e}_1, \tilde{e}_4) + \tilde{R}(\tilde{e}_2, \tilde{e}_3, \tilde{e}_2, \tilde{e}_3) \\ + \tilde{R}(\tilde{e}_2, \tilde{e}_4, \tilde{e}_2, \tilde{e}_4) - 2\tilde{R}(\tilde{e}_1, \tilde{e}_2, \tilde{e}_1, \tilde{e}_4) \geq 0 \end{aligned}$$

Therefore, (3) holds. □

Now we present a classification result about homeomorphic sphere.

Proposition 3.1.4. *Let (M^n, g) be closed with $K_g^{\mathbb{C}} \geq 0$. If M is homeomorphic to a sphere, then the Ricci flow $g(t)$ with $g(0) = g$ has $K_{g(t)}^{\mathbb{C}} > 0$ for all $t > 0$.*

Proof. Since M is a sphere, the metric is irreducible and neither Kähler nor quaternion-Kähler.

If (M, g) is a locally symmetric space, it is symmetric as it is connected and hence homogeneous. Homogeneous simply-connected rational cohomology spheres are all classified (see [4]). These results give a list of pairs (G, H) such that G/H is homeomorphic to a sphere. One can compare the list of classification of symmetric space to see that $SO(n+1)/SO(n)$ is the only possiblity which is a round sphere. So, $K_g^{\mathbb{C}} > 0$. Since the positivity of complex sectional curvature is preserved under Ricci flow (see Proposition 7.28 in [29]). In this case, it is done.

If it is not locally symmetric, we deduce that g as well as $g(t)$ has $SO(n)$ holonomy by using the classification result of Berger [15]. We now prove that $K_{g(t)}^{\mathbb{C}} > 0$ for all $t > 0$. We follow the steps in Proposition 7 of [27]. Let $t' > 0$ and $g' = g(t')$. Noticed that since $K^{\mathbb{C}} \geq 0$ is preserved under Ricci flow (see Proposition 7.28 in [29]), by Proposition (3.1.3), we have

$$\begin{aligned} R_{g'}(e_1, e_3, e_1, e_3) + \lambda^2 R_{g'}(e_1, e_4, e_1, e_4) + \mu^2 R_{g'}(e_2, e_3, e_2, e_3) \\ + \lambda^2\mu^2 R_{g'}(e_2, e_4, e_2, e_4) - 2\mu\lambda R_{g'}(e_1, e_2, e_3, e_4) \geq 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\mu, \lambda \in [-1, 1]$.
Now, it suffices to show that

$$\begin{aligned} R_{g'}(e_1, e_3, e_1, e_3) + \lambda^2 R_{g'}(e_1, e_4, e_1, e_4) + \mu^2 R_{g'}(e_2, e_3, e_2, e_3) \\ + \lambda^2\mu^2 R_{g'}(e_2, e_4, e_2, e_4) - 2\mu\lambda R_{g'}(e_1, e_2, e_3, e_4) > 0 \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\mu, \lambda \in [-1, 1]$. Suppose it is not true, there exists a orthonormal frame $\{e_1, e_2, e_3, e_4\}$ in T_pM with respect to g' and satisfies

$$\begin{aligned} &R_{g'}(e_1,e_3,e_1,e_3)+\lambda^2 R_{g'}(e_1,e_4,e_1,e_4)+\mu^2 R_{g'}(e_2,e_3,e_2,e_3)\\ &\qquad +\lambda^2\mu^2 R_{g'}(e_2,e_4,e_2,e_4)-2\mu\lambda R_{g'}(e_1,e_2,e_3,e_4)=0. \end{aligned}$$

Since the holonomy group is $SO(n)$, the manifold is not flat. Hence we can find a point $q \in M$ and an orthonormal frame $\{v_1, v_2\} \subset T_qM$ such that

$$R_{g'}(v_1,v_2,v_1,v_2)>0.$$

Let $\gamma : [0,1] \to M$ be a piecewise smooth path from p to q. Since $Hol^0(M,g') = SO(n)$, we can find a loop $\sigma : [0,1] \to M$ at p such that $v_1 = P_\gamma \circ P_\sigma\ e_1$ and $v_2 = P_\gamma \circ P_\sigma\ e_2$ (Here, P_γ denotes the parallel transport along γ with respect to the metric g'). By using the Proposition 9 in [27], we know that the equality is invariant under parallel transport. So

$$\begin{aligned} &R_{g'}(v_1,v_3,v_1,v_3)+\lambda^2 R_{g'}(v_1,v_4,v_1,v_4)+\mu^2 R_{g'}(v_2,v_3,v_2,v_3)\\ &\qquad +\lambda^2\mu^2 R_{g'}(v_2,v_4,v_2,v_4)-2\mu\lambda R_{g'}(v_1,v_2,v_3,v_4)=0 \end{aligned} \tag{3.2}$$

where $v_3, v_4 \in T_qM$ defined by $v_i = P_{\gamma\circ\sigma}e_i$, $i = 3, 4$. Similarly, we can show that

$$\begin{aligned} &R_{g'}(v_1,v_2,v_1,v_2)+\lambda^2 R_{g'}(v_2,v_4,v_2,v_4)+\mu^2 R_{g'}(v_1,v_3,v_1,v_3)\\ &\qquad +\lambda^2\mu^2 R_{g'}(v_3,v_4,v_3,v_4)-2\mu\lambda R_{g'}(v_2,v_3,v_1,v_4)=0 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &R_{g'}(v_2,v_3,v_2,v_3)+\lambda^2 R_{g'}(v_3,v_4,v_3,v_4)+\mu^2 R_{g'}(v_2,v_1,v_2,v_1)\\ &\qquad +\lambda^2\mu^2 R_{g'}(v_1,v_4,v_1,v_4)-2\mu\lambda R_{g'}(v_3,v_1,v_2,v_4)=0. \end{aligned} \tag{3.4}$$

Sum up (3.2), (3.3) and (3.4). This yields

$$\begin{aligned} &[R_{g'}(v_1,v_2,v_1,v_2)+R_{g'}(v_1,v_3,v_1,v_3)+R_{g'}(v_2,v_3,v_2,v_3)]\\ &+\lambda^2[R_{g'}(v_1,v_4,v_1,v_4)+R_{g'}(v_2,v_4,v_2,v_4)+R_{g'}(v_4,v_3,v_4,v_3)]\\ &=0. \end{aligned}$$

Since (M, g') has nonnegative sectional curvature, it follows that

$$R_{g'}(v_1, v_2, v_1, v_2) = 0$$

which contradicts with our assumption.

$\square$

3.2 Cheeger-Gromoll convex exhaustion.

Let (M, g) be a nonnegatively curved open manifold. A ray is a unit speed geodesic $\gamma : [0, +\infty) \to M$ such that $\gamma|_{[0,s]}$ is minimizing geodesic for any $s > 0$. Fix $o \in M$, let

$$Q = \{\gamma : [0, +\infty) \to M : \gamma \text{ is a ray with } \gamma(0) = o\}.$$

Consider the busemann function b of M.

$$b(p) = \sup_{\gamma \in Q}\{\lim_{s\to\infty}(s - d_g(\gamma(s), p))\}.$$

In case of non-negatively curved open manifold, Cheeger and Gromoll (see [13]) show that b is a convex function, that is for any geodesic $c(s) \in M$ the function $b \circ c(s)$ is convex function on $\mathbb{R}$.

Throughout the chapter, we will make use of the family of sublevel set

$$C_l \doteq b^{-1}((-\infty, l])$$

to construct a sequence of Ricci flow and sub-converge to a solution on M. The following properties of C_l will be used in this paper.

Proposition 3.2.1. *If (M, g) has nonnegative curvature with C_l constructed above, then C_l has the following properties. (see section 1 in [13])*

(1): Each C_l is a totaly convex compact set,

(2): dim $C_l = n$ *for all* $l > 0$, $\cup_{l>0} C_l = M$,

(3): $s < l$ *implies* $C_s \subset C_l$ *and* $C_s = \{x \in C_l : d_g(x, \partial C_l) \geq l - s\}$,

(4): each $C_l, l > 0$, *has the structure of an embedded submanifold of* M *with smooth totally geodesic interior and possibly non-smooth boundary.*

Furthermore, if (M, g) has positive sectional curvature, by local smoothing procedure, one can modify the Busemann function to obtain a smooth function β.

Theorem 3.2.2. *(see [31]) If* (M, g) *is an open manifold with* $K_g > 0$, *then there exists a smooth proper strictly convex function* $\beta : M \to [0, \infty[$.

3.3 Approximating sequence for the initial condition.

Let (M, g) be an open manifold with $K_g > 0$. On M, we consider the function β. Since β is proper and bounded below, global minimum is attained. Without loss of generality, we assume the global minimum is 0. Furthermore, since β is strictly convex, we have $\beta^{-1}(0) = \{p_0\}$. Since β is a smooth convex function, hence the sublevel set $C_i = \{x \in M : \beta(x) \leq i\}$ is a convex set with smooth boundary for all $i > 0$.

Our goal here is to construct a pointed sequence of closed manifolds converging to (M, g, p_0) in the following sense.

Definition 3.3.1. *(Cheeger-Gromov convergence). Let* (M_i^n, g_i, p_i) *be a sequence of complete manifolds. We say* (M_i, g_i, p_i) *converges to the pointed Riemannian manifold* $(M_\infty, g_\infty, p_\infty)$ *if there exists*

(1): a collection of $\{U_i\}_{i\geq 1}$ *of compact sets with* $U_i \subset U_{i+1}$, $\cup_{i\geq 1} U_i = M_\infty$ *and* $p_\infty \in int(U_i)$ *for all* i

(2): $\phi_i : U_i \to M_i$ diffeomorphisms onto their image, with $\phi_i(p_\infty) = p_i$

such that $\phi_i^* g_i \to g_\infty$ smoothly on compact set of M_∞, that is

$$|\nabla^m(\phi_i^* g_i - g_\infty)| \to 0, \quad as \quad i \to \infty \quad on \quad K, \quad \forall m \in \mathbb{N}$$

for every compact set $K \subset M_\infty$. Here the norm and ∇ are computed with respect to any fixed background metric.

A sequence of complete evolving manifolds $(M_i, g_i(t), p_i)_{t\in I}$ converges to a pointed evolving manifold $(M_\infty, g_\infty(t), p_\infty)_{t\in I}$ if we have (1),(2) as before such that $\phi_i^*(t) \to g_\infty(t)$ smoothly on compact subsets of $M_\infty \times I$.

The first attempt would be to consider the double $D(C_i)$ of C_i. However, $D(C_i)$ may not be a smooth manifold. So we try to modify the metric near the boundary ∂C_i to form cylindrical end so that the gluing is well defined.

Proposition 3.3.2. *Let (M, g) be an open manifold with $K_g^{\mathbb{C}} > 0$ and soul point p_0. Then there exists a collection $(M_i, g_i, p_0)_{i\geq 1}$ of smooth closed n-dimensional pointed manifolds with $K_{g_i}^{\mathbb{C}} > 0$ satisfying*

$$(M_i, g_i, p_0) \to (M, g, p_0) \quad as \quad i \to \infty$$

in the sense of the smooth Cheeger-Gromov convergence .

Proof. Let $s > 0$ be fixed and small. For each fixed C_i, choose φ_i such that

$$\begin{cases} \text{(a)} \ \ \varphi_i \text{ is smooth on } (-\infty, i) \text{ and continuous at } i, \\ \text{(b)} \ \ \varphi_i \equiv 0 \text{ on}(-\infty, i-s] \text{ and } \varphi_i(i) = 1, \\ \text{(c)} \ \ \varphi_i', \varphi_i'' \text{ are positive on } (i-s, i), \\ \text{(d)} \ \ \text{and the inverse of } \varphi_i, \varphi_i^{-1}, \text{ has all left derivatives vanishing at } i. \end{cases}$$

Take $u_i = \varphi_i \circ \beta$, and put

$$G_i = \{(x, u_i(x)) : x \in C_i\}$$
$$\tilde{G}_i = \{(x, 2 - u_i(x)) : x \in C_i\}$$

(d) ensures that they paste smoothly together to a C^∞ closed hypersurface $D(G_i) = G_i \cup \tilde{G}_i \subset M \times \mathbb{R}$, where $M \times \mathbb{R}$ has nonnegative complex sectional curvature. We now claim that $D(G_i) = (M_i, g_i)$ has nonnegative complex sectional curvature. First we observe that u_i is convex function on C_i. Since for any vector field X on M,

$$\nabla^2 u_i(X, X) = \langle \nabla_X \nabla u_i, X\rangle = \varphi_i'' \cdot (\nabla_X \beta)^2 + \varphi_i' \cdot \nabla^2 \beta(X, X) \geq 0.$$

Noted that the submanifolds G_i and $\tilde{G}_i$ are isometric. So it suffices to prove that G_i has nonnegative complex sectional curvature.

For simplicity we denote u_i by f. Let $q = (p, f(p)) \in int(G_i)$, consider $T_q G_i$. Let $\{x^j\}$ be local chart at p. Then $(x^1, ..., x^n, t)$ gives a chart at $q \in M \times \mathbb{R}$. $\{e_j\}_{j=1}^n = \left\{ \frac{\partial}{\partial x^j} + \frac{\partial f}{\partial x^j}\frac{\partial}{\partial t} \right\}_{j=1}^n$ forms a basis for $T_q G_i$. Let $\bar{\nabla}$ be the connection on $M \times \mathbb{R}$ induced by the product metric. By Gauss Codazzi equations,

$$R(X, Y, \bar{X}, \bar{Y}) = \bar{R}(X, Y, \bar{X}, \bar{Y}) + \langle B(X, \bar{X}), B(Y, \bar{Y})\rangle - |B(X, \bar{Y})|^2.$$

Here $\bar{R}m$ is the curvature tensor on $M \times \mathbb{R}$ and $B(\cdot, \cdot)$ is the second fundamental form which are extended complex linearly. So it suffices to show that for each i, j

$$\langle \bar{\nabla}_{e_i} e_i{}^\perp, \bar{\nabla}_{e_j} e_j{}^\perp\rangle - \langle \bar{\nabla}_{e_i} e_j{}^\perp, \bar{\nabla}_{e_i} e_j{}^\perp\rangle \geq 0$$

Assume $\left\{ \frac{\partial}{\partial x^j} \right\}_{j=1}^n$ is an orthonormal basis at p, then we obtain

$$\bar{\nabla}_{e_i} e_j = \frac{\partial^2 f}{\partial x^i \partial x^j}\frac{\partial}{\partial t} \quad \text{at} \quad q = (p, f(p)).$$

Let N be the normal vector field on G_i , and $N' = \langle N, \frac{\partial}{\partial t}\rangle$. It implies

$$\bar{\nabla}_i e_j{}^\perp = N' \frac{\partial^2 f}{\partial x^i \partial x^j}\frac{\partial}{\partial t} \qquad , \forall i, j = 1, 2...n,$$

Since f is convex function, we get

$$\langle \bar{\nabla}_{e_i} e_i{}^\perp, \bar{\nabla}_{e_j} e_j{}^\perp\rangle - \langle \bar{\nabla}_{e_i} e_j{}^\perp, \bar{\nabla}_{e_i} e_j{}^\perp\rangle = (f_{ii} f_{jj} - f_{ij}^2)(N')^2 \geq 0$$

If $q \in \partial G_i \subset D(G_i)$, let q_j be a sequence of point in $int(G_i)$ such that $q_j \to q$ as $j \to \infty$. From above, we have $K^{\mathbb{C}}(q_j) \geq 0$. By taking limit, we deduce that $K^{\mathbb{C}}(q) \geq 0$. Noticed that $C_{i-\epsilon}$ can be seen as subset of M_i for all $i > 0$, which implies that (M_i, g_i, p_0) converges to (M, g, p_0) in the Cheeger-Gromov sense. Since (M_i, g_i, p_0) are closed manifolds, we can use the short time existence of the Ricci flow on M_i (see [26]), and choose $t_i > 0$ small enough that $(M_i, g_i(t_i), p_0)$ still converges to (M, g, p_0). By Thm 2.5 in [13], we know that $C_{i-\epsilon}$ is homeomorphic to a disc. Hence, M_i is a topological sphere. We can employ Proposition (3.1.4) to conclude that $K^{\mathbb{C}}_{g_i(t_i)} > 0$. Thus $g_i(t_i)$ is a solution of our problem. □

3.4 Ricci flow on the approximating sequence.

Consider (M_i, g_i, p_0) the sequence of closed, positively curved manifolds obtained from above. For each i, we construct a Ricci flow $(M_i, g_i(t))$ defined on a maximal time interval $[0, T_i)$ with $g_i(0) = g_i$.

The first difficulty to address is that the curvature of g_i may tend to infinity as $i \to \infty$. It maybe happen that $T_i \to 0$ as $i \to \infty$. So our next concern is to prove that T_i admit a uniform lower bound independent of i. We estimate it by considering the volume growth of unit balls around p_0. For such estimate, we make a strong use of the following theorem.

Theorem 3.4.1. *(Petrunin,[1]) Let (M^n, g) be a complete manifold with $K_g \geq -1$. Then for any p in M*

$$\int_{B_g(p,1)} scal_g \, dV_g \leq C_n,$$

for some constant C_n depending on the dimension only.

Proposition 3.4.2. *Let (M, g) and (M_i, g_i, p_0) be as in Proposition 3.3.2. Then there exists a constant $\tau > 0$, depending on n, and $V_0 = vol_g(B_g(p_0, 1))$, such that the Ricci flows $(M, g_i(t))$ with $g_i(0) = g_i$ are defined on $[0, \tau]$, and satisfy $K^{\mathbb{C}}_{g_i(t)} > 0$ for all $t \in [0, \tau]$.*

Proof. For each i, (M_i, g_i) is a closed n-manifold. So there exists some $T_i > 0$ and a unique maximal Ricci flow $(M_i, g_i(t))$ defined on $[0, T_i)$ with $g_i(0) = g_i$. Moreover , $K^{\mathbb{C}}_{g_i(t)} > 0$ for all $t \in [0, T_i)$ as positive complex sectional curvature is preserved under Ricci flow. It remains to show the uniform lower bound for the lifespan.

Since $Ric_{g_i(t)} > 0$,the metric is shrinking which implies that $B_{g_i(0)}(p_0, 1) \subset B_{g_i(t)}(p_0, 1)$. Using the evolution equation of volume form, one can deduce that

$$\frac{\partial}{\partial t} vol_{g_i(t)}(B_{g_i(0)}(p_0, 1)) = -\int_{B_{g_i(0)}(p_0,1)} scal_{g_i(t)}\, dV_{g_i(t)}$$

Since $scal_{g_i(t)} > 0$ and $B_{g_i(0)}(p_0, 1) \subset B_{g_i(t)}(p_0, 1)$,

$$\frac{\partial}{\partial t} vol_{g_i(t)}(B_{g_i(0)}(p_0, 1)) \geq -\int_{B_{g_i(t)}(p_0,1)} scal_{g_i(t)}\, dV_{g_i(t)} \geq -C_n \tag{3.5}$$

The last inequality follows from Theorem 3.4.1. Hence,

$$vol_{g_i(t)}(B_{g_i(0)}(p_0, 1)) - vol_{g_i(0)}(B_{g_i(0)}(p_0, 1)) \geq -C_n t \geq -C_n T_i$$

So,

$$T_i \geq \frac{1}{C_n} \left[vol_{g_i(0)}(B_{g_i(0)}(p_0, 1)) - vol_{g_i(t)}(B_{g_i(0)}(p_0, 1)) \right]$$

On the other hand, as M_i is closed with $K^{\mathbb{C}}_{g_i} > 0$, the normalized Ricci flow converge to a metric of positive constant sectional curvature as time goes to infinity (see [28]). Thus, the volume of $(M_i, g_i(t))$ vanishes completely at the maximal time T_i . So

$$T_i \geq \frac{1}{C_n} vol_{g_i(0)}(B_{g_i(0)}(p_0, 1)) \to \frac{1}{C_n} vol_g(B_g(p_0, 1)) = 2\tau.$$

□

As a consequence, we obtain a uniform lower bound for the volume of unit balls centered at the soul point.

Corollary 3.4.3. *For the sequence of pointed Ricci flows* $(M_i, g_i(t), p_0)_{t\in[0,\tau]}$ *from Proposition 3.4.2, we can find a constant* $v_0 = v_0(n, V_0)$ *satisfying*

$$vol_{g_i(t)}(B_{g_i(t)}(p_0, 1)) \geq v_0 > 0, \quad \forall t \in [0, \tau].$$

Proof. Using again (3.5) and $t \leq \tau = \dfrac{V_0}{2C_n}$, we obtain

$$\begin{aligned} vol_{g_i(t)}(B_{g_i(t)}(p_0, 1)) &\geq vol_{g_i(t)}(B_{g_i(0)}(p_0, 1)) \\ &\geq vol_{g_i(0)}(B_{g_i(0)}(p_0, 1)) - C_n t \\ &\geq \frac{3}{4}V_0 - C_n\tau \\ &= \frac{V_0}{4} = v_0 > 0. \end{aligned}$$

□

3.5 Interior curvature estimates around the soul point.

In order to get a limit Ricci flow solution from the sequence $(M_i, g_i(t))$, the first step is to obtain uniform curvature estimates independent of i. In this section, we will show the curvature estimate around the soul point p_0.

Lemma 3.5.1. *Let* (M^n, g) *be an open manifold with* $K_g^{\mathbb{C}} \geq 0$*, then*

$$\{u : Rm(u \wedge v) = 0 \text{ , for all } v \in \mathfrak{T}(M)\} = \{u : Rm(u, v, u, v) = 0 \text{ , for all } v \in \mathfrak{T}(M)\}.$$

Proof. Clearly, $\{u : Rm(u \wedge v) = 0, \text{ for all } v \in \mathfrak{T}(M)\} \subset \{u : Rm(u, v, u, v) = 0, \text{ for all } v \in \mathfrak{T}(M)\}$. It remains to show the opposite direction. Let u be an element in $\{u : Rm(u, v, u, v) = 0, \text{ for all } v\}$.

If $n = 3$, since $K \geq 0$ will imply $Rm \geq 0$. For any $v \in T_pM$ and $\phi \in \bigwedge^2(T_pM)$, by considering

$$f(t) = \langle Rm_g\left(u \wedge v + t\phi\right), u \wedge v + t\phi \rangle \geq 0 \text{ , } t \in \mathbb{R}$$

which attains minimum at $t = 0$. $f'(0) = 0$ implies that $Rm_g(u \wedge v) = 0$.

If $n \geq 4$, by Proposition (3.1.3), we have for all orthonormal base $\{e_k\}_{k=1}^n$, where $e_1 = u/||u||$ and $\lambda, \mu \in [-1, 1]$,

$$R_{1313} + \mu^2 R_{1414} + \lambda^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq 0$$

If we let

$$F(\lambda, \mu) = \lambda^2 R_{2323} - 2\lambda\mu R_{1234} + \lambda^2 \mu^2 R_{2424}, \ \lambda, \mu \in [-1, 1].$$

Since we know $F \geq 0$ and attains minimum at $(0, 0)$, $\nabla^2 F \geq 0$ at $(0, 0)$ implying $R_{0123} = 0$. As e_2, e_3, e_4 can be arbitarily chosen, we deduce that for any $i, j, k > 1$ distinct,

$$\langle Rm_g(e_1 \wedge e_i), e_j \wedge e_k \rangle = 0.$$

For any $i, j \in \{2, 3, ...n\}$ distinct, we let $H(t) = Rm(e_1 + te_j, e_i, e_1 + te_j, e_i)$, $t \in \mathbb{R}$. Since $K \geq 0$, we have $H(t) \geq 0$ and attains minimum at $t = 0$. So $H'(0) = 0$ which imply

$$\langle Rm_g(e_1 \wedge e_i), e_i \wedge e_j \rangle = 0.$$

Similarly, we also have

$$\langle Rm_g(e_1 \wedge e_i), e_1 \wedge e_j \rangle = 0.$$

Thus, $Rm_g(u \wedge v) = 0$ for all v at p.

□

Lemma 3.5.2. *Let $(M^n, g(t)), t \in [0, T]$ be an complete solution of Ricci flow with bounded curvature. Then for each $t \in [0, T]$, $N(t) = \{u : Rm(t)(u \wedge v) = 0,$ for all $v \in \mathfrak{T}(M)\}$ is invarient under parallel translation.*

Proof. Fix $t \in [0, T]$, $u \in N(t)$, we would like to show that for any vector field X, $\nabla_X u \in N(t)$. By Lemma (3.5.1), it suffices to show that $Rm(\nabla_i u, v, \nabla_i u, v) = 0$

for all v and coordinate vector field $\partial_i = \dfrac{\partial}{\partial x^i}$.

At $p \in M$, $Rm(u,v,u,v) = 0$, extend $u(p), v(p)$ to U, V by parallel transport along the integral curve $c(s)$ of ∂_i emanating from p. By strong maximum principle in [27], we have $Rm(U,V,U,V)(c(s)) = 0$ for all $s \in \mathbb{R}$. Taking first and second derivatives with respect to s and evaluate at p, this yields

$$(\nabla_i Rm)(u,v,u,v) = 0 \quad \text{and} \quad (\nabla_i \nabla_i Rm)(u,v,u,v) = 0, \text{ at } p.$$

As p is arbitarily chosen, the equalities hold on M. Taking derivative on the first equation, we have

$$\begin{aligned} 0 &= \partial_i[(\nabla_i Rm)(u,v,u,v)] \\ &= (\nabla_i \nabla_i Rm)(u,v,u,v) + 2(\nabla_i Rm)(\nabla_i u,v,u,v) + 2(\nabla_i Rm)(u,\nabla_i v,u,v) \end{aligned}$$

We obtain

$$(\nabla_i Rm)(\nabla_i u,v,u,v) + (\nabla_i Rm)(u,\nabla_i v,u,v) = 0. \tag{3.6}$$

Consider the first term.

$$\begin{aligned} (\nabla_i Rm)(\nabla_i u,v,u,v) &= \partial_i[(Rm)(\nabla_i u,v,u,v)] - Rm(\nabla_i \nabla_i u,v,u,v) \\ &\quad - Rm(\nabla_i u,\nabla_i v,u,v) - Rm(\nabla_i u,v,\nabla_i u,v) - Rm(\nabla_i u,v,u,\nabla_i v) \\ &= -Rm(\nabla_i u,v,\nabla_i u,v). \end{aligned}$$

Doing similar step on the second term, together with (3.6), we deduce that

$$Rm(\nabla_i u,v,\nabla_i u,v) = 0.$$

□

Lemma 3.5.3. *Let $(M^n, g(t)), t \in (-\infty, 0]$ be an open non-flat ancient solution of the Ricci flow. Assume further that $g(t)$ has bounded curvature operator, and that $K^{\mathbb{C}}_{g(t)} \geq 0$. Then for all $t \in (-\infty, 0]$, we have*

$$v(t) = \lim_{r\to\infty} \frac{vol_{g(t)}(B_{g(t)}(\cdot, r))}{r^n} = 0.$$

Proof. We first claim that $v(t)$ is non-increasing on time t. Since the curvature is bounded, by Lemma 8.3 in [12], there exists $C > 0$ such that for all $p, x \in M$,

$$\frac{d}{dt} d_t(p, x) \geq -C$$

which implies for $s > 0$,

$$d_{s+t}(p, x) \geq d_t(p, x) - Cs. \tag{3.7}$$

Thus, $B_{t+s}(p, r) \subset B_t(p, Cs + r)$, which gives

$$vol_{g(t+s)}(B_{t+s}(p, r)) \leq vol_{g(t+s)}(B_t(p, Cs + r)) \leq vol_{g(t)}(B_t(p, Cs + r)).$$

The last inequality is due to the fact that the metric is shrinking. So, for all $s > 0$,

$$\lim_{r\to\infty} \frac{vol_{g(t+s)}(B_{t+s}(p, r))}{r^n} \leq \lim_{r\to\infty} \frac{vol_{g(t)}(B_t(p, Cs + r))}{(Cs + r)^n} \cdot \left(\frac{Cs + r}{r}\right)^n = v(t)$$

which implies $v(t)$ is non-increasing on time t.

We now prove the lemma by induction on dimension n. When $n = 2$, if the statment is false, then there exists $t_0 \leq 0$ such that

$$v(t_0) = \lim_{r\to\infty} \frac{vol_{g(t_0)}(B_{g(t_0)}(\cdot, r))}{r^n} > 0.$$

Combining with the fact that $v(t)$ is non-increasing, $v(t) \geq v(t_0) > 0$, for all $t \leq t_0$. Thus, it is a κ-solution. By Corollary 11.3 in [12], there are no noncompact κ-solution in dimension 2. Hence, we are done in this case.

Assume the statement holds in dimension $n-1$, where $n \geq 3$. If the statement is false, there exists t_0 such that for all $t \leq t_0$,

$$v(t) \geq v(t_0) > 0. \tag{3.8}$$

Without loss of generality, we assume $t_0 = 0$. Now, we consider the following 3 cases.

Case A. $\mathrm{ASCR}(0) = +\infty$

Case B. $\mathrm{ASCR}(0) \in (0, \infty)$

Case C. $\mathrm{ASCR}(0) = 0$

where $\mathrm{ASCR}(t) = \limsup_{d_t(x,p)\to\infty} d_t^2(x,p)R(x,t)$, p is a fixed point on M.

Case A.

If $\mathrm{ASCR}(0) = +\infty$, By Lemma 22.2 in [25], there exists a sequence of points $\{x_i\}_{i=1}^{\infty}$ with $d_0(x_i,p) \to \infty$ and radius $r_i > 0$ such that $R(x_i,0)r_i^2 \to \infty$, $d_0(x_i,p)/r_i \to \infty$, and

$$R(x,0) \leq 2R(x_i,0) \ \ , \forall\, x \in B_0(x_i,r_i).$$

Let

$$g_i(t) = R(x_i,0)g\left(\frac{t}{R(x_i,0)}\right) \ \ , \ t \in (-\infty, 0].$$

The assumption (3.8) implies that $inj_{g_i(0)}(x_i) \geq \delta$ for some $\delta > 0$. By trace Harnack inequality in [30], we have $\frac{\partial}{\partial t}R \geq 0$. Thus we have for all $x \in B_{g_i(0)}(x_i, \sqrt{R(x_i,0)}r_i)$, $t \in (-\infty, 0]$,

$$R_{g_i}(x,t) \leq R_{g_i}(x,0) \leq 2.$$

Applying Hamilton's Cheeger-Gromov-type compactness theorem, we conclude

$$\left(B_{g_i(0)}(x_i, \sqrt{R(x_i,0)}r_i), g_i(t), x_i\right) \to (M_\infty^n, g_\infty(t), x_\infty)$$

where $t \leq 0$ and $R_{g_\infty}(x_\infty, 0) = 1$. The limit is a complete non-compact solution with $R_{g_\infty}(x,t) \leq 2$.

Since the metric is shrinking, by the construction of $\{x_i\}$, we get

$$R(x_k, 0)d^2(x_i, p, s) \geq R(x_k, 0)d^2(x_i, p, 0) \to +\infty$$

as k tends to infinity, for any $s \leq 0$. Thus, by Proposition 6.1.2 in [7], we know that $(M^n_\infty, g_\infty(s), x_\infty)$ splits off a line for each $s \leq 0$.

We now calim that there exists $t' \leq 0$ such that the limit solution splits as product on $(-\infty, t']$. By above, we know that at $t = 0$

$$(M^n_\infty, g_\infty(0), x_\infty) = (\mathbb{E}_1 \times W_1^{n-1},\ du_1^2 + g_{W1})$$

where $\mathbb{E}_1$ is a copy of $\mathbb{R}$. We denote 0 by s_1. If the splitting holds for all $t < s_1$, then we are done by taking $t' = s_1$.

Otherwise, there exists $s_2 < 0$ such that at $t = s_2$, $(M_\infty, g_\infty(s_2))$ doesn't split off $\mathbb{E}_1$. But since it must split off a line, we have

$$(M^n_\infty, g_\infty(s_2), x_\infty) = (\mathbb{E}_2 \times W_2^{n-1},\ du_2^2 + g_{W_2})$$

where $\mathbb{E}_2$ is another copy of $\mathbb{R}$. By Theorem 1.1 in [6], we know that $(M_\infty, g_\infty(s_1))$ must splits off $\mathbb{E}_2$. That is

$$(M^n_\infty, g_\infty(0), x_\infty) = (\mathbb{E}_1 \times \mathbb{E}_2 \times W_{12}^{n-1},\ du_1^2 + du_2^2 + g_{W_{12}})$$

If the splitting holds for all $t < s_2$, then we take $t' = s_2$. Otherwise repeat the above procedures to get s_3. But as the dimension is finite, the process can only be iterated for finitely many times. So we can let t' be the last s_k. And we can conclude that

$$(M^n_\infty, g_\infty(t), x_\infty) = (\mathbb{R} \times W^{n-1},\ du^2 + g_W)\ , t \in (-\infty, t'].$$

In particularly, the $\mathbb{R}$ component comes from the splitting line at time s_k.

Now we claim that $(W, g_W(t'))$ has positive asymptotic volume ratio. By volume comaprsion theorem, we have for any $r > 0$, and $i \in \mathbb{N}$,

$$\begin{aligned}\frac{Vol_{g_i(t')}B_{g_i(t')}(x,r)}{r^n} &= \frac{Vol_{g(t'/R(x_i,0))}B_{g(t'/R(x_i,0))}(x,\sqrt{R(x_i,0)}r)}{(\sqrt{R(x_i,0)}r)^n}\\ &\geq v(t'/R(x_i,0)) \geq v(0) > 0.\end{aligned}$$

Passing to limit, we obtain $\frac{Vol_{g_\infty(t')}B_{g_\infty(t')}(x_i,r)}{r^n} \geq v(0) > 0$, for all $r > 0$. Denote $x_\infty = (0, x_W)$, for product metric, we have

$$\begin{aligned}B_{g_\infty(t')}(x_\infty, r) &\subset (-r,r) \times B_{g_W(t')}(x_W, r),\\ Vol_{g_\infty(t')}(B_{g_{\infty(t')}}(x_\infty, r)) &\leq\ 2rVol_{g_W(t')}(B_{g_W(t')}(x_W, r)).\end{aligned}$$

Hence,

$$\frac{Vol_{g_W(t')}(B_{g_W(t')}(x_W,r))}{r^{n-1}} \geq \frac{Vol_{g_\infty(t')}(B_{g_{\infty(t')}}(x_\infty,r))}{2r^n} > 0$$

which contradicts with the induction hypothesis.

Case B.

If ASCR(0) $\in (0,\infty)$, by the definition of ASCR, there exists a sequence of points $x_i \in M$ such that as $i \to \infty$,

$$d_{g(0)}(x_i,p) \to \infty, \quad R(x_i,0)d^2_{g(0)}(x_i,p) \to \text{ASCR}(0).$$

Let b, B be two real numbers such that $0 < b < \sqrt{\text{ASCR}(0)} < B < \infty$. Define the rescaled solution $\{(M, g_i(t))\}_{i\in\mathbb{N}}$ with $g_i(t) = R(x_i,0)g\left(\frac{t}{R(x_i,0)}\right)$. We have as $i \to \infty$,

$$d_{g_i(0)}(x_i,p) = \sqrt{R(x_i,0)}d_{g(0)}(x_i,p) \to \sqrt{\text{ASCR}(0)} \in (b,B).$$

Let $N_i(b,B) = B_{g_i(0)}(p,B) \setminus \overline{B_{g_i(0)}(p,b)}$. By trace Harnack inequality in [30], we have the curvature bound

$$R_{g_i}(x,t) \leq R_{g_i}(x,0) \leq \frac{2ASCR(0)}{d^2_{g_i(0)}(x,p)} \leq C(b, ASCR(0))$$

for all $x \in N_i(b, B)$, and $t \leq 0$. Again by the assumption 3.8, we have $inj_{g_i(0)}(x_i) \geq \delta > 0$. Applying the local compactness theorem, we obtain that for a subsequence,

$$(N_i(b, B), g_i(t), x_i) \to (N_\infty(b, B), g_\infty(t), x_\infty)$$

as $i \to \infty$. On the other hand, as $g_i(0) = R(x_i, 0)g(0)$, $R(x_i, 0) \to 0$ as $i \to \infty$, and $K(g_i(0)) \geq 0$, by Theorem I.26 in [3], we have that

$$(M^n, g_i(0), p) \to (CW, d_\infty, p_\infty)$$

converges in the pointed Gromov-Hausdorff topology as $i \to \infty$, where

$$CW = (\,[0, \infty) \times W)/(\{0\} \times W).$$

By changing the fixed point $p \in M$ to a sequence of fixed point $\{y_k\}$ which is uniformly bounded distance away from each others. Then the corresponding Gromov-Hausdorff limit is isometric to the original limit which gives a smooth $(n-1)$ manifold structure to W. And there exists a Riemannian metric g_W on W such that (CW, d_∞) has a Riemannian metric given by

$$g_\infty(0) = dr^2 + r^2 g_W. \tag{3.9}$$

At $p \in CW$, let $\{y^j\}_{j=1}^{n-1}$ be the local coordinates on W. We further assume $\{\partial_j = \frac{\partial}{\partial y^j}\}_{j=1}^{n-1}$ is normal coordinate at p. Since metric is in form of (3.9), we have

$$\langle Rm_{g_\infty(0)}\left(\partial_r \wedge \partial_j\right), \partial_r \wedge \partial_j\rangle = 0, \ \forall\ j = 1, 2, ...n-1.$$

Since $K^{\mathbb{C}}_{g_\infty(t)} \geq 0$, as shown in the proof of Lemma (3.5.1), we deduce that for all j, $Rm_{g_\infty(0)}\left(\partial_r \wedge \partial_j\right) = 0$ at p. By Lemma (3.5.2), it implies

$$Rm(\nabla_j \partial_r \wedge \partial_i) = 0 \text{ for all } j, \text{ at } p.$$

On the other hand, (3.9) implies $\nabla_j \partial_r = \frac{1}{r}\partial_j$. Thus, $Rm(\partial_j \wedge \partial_i) = 0$ for any i, j which means $Rm_{g_\infty(0)} = 0$ at p. As p is arbitary point on CW. It is flat which contradicts with the fact that $R_{g_\infty(0)}(x_\infty, 0) = 1$.

Case C.

If ASCR(0) $= 0$, we have $\limsup_{d(x,0)\to} R(x) = 0$. By (3.7), and Harnack inequality in [30], we infer that for each $t \leq 0$ and each $x \in M$,

$$0 \leq d^2(x,t)R(x,t) \leq [d(x,0) - Ct]^2 R(x,0).$$

It implies that ASCR$(t) = 0$ for any $t \leq 0$. By the result of Petrunin and Tuschmann (see Theorem B in [20]), we have for each $t \leq 0$, the universal cover of $(M^n, g(t))$ is isometric to $\mathbb{R}^{n-2} \times (\Sigma, g_\Sigma)$ and that (Σ, g_Σ) has ASCR$= 0$. Using similar arguement as in case A, the universal cover of $(M^n, g(t))$ is isometric to $\mathbb{R}^{n-2} \times (\Sigma, g_\Sigma(t))$ at which $(\Sigma, g_\Sigma(t))$ is a κ-solution. But the only two dimension κ-solution is round sphere in which $\mathbb{R}^{n-2} \times (\Sigma, g_\Sigma(0))$ is not possible to have ASCR$(0) = 0$. So this case can be ruled out. □

Proposition 3.5.4. *Consider the Ricci flows $(M_i, g_i(t))$ with $t \in [0,\tau]$, coming from Proposition 3.4.2. For any $D > 0$, there exists a constant $C_D > 0$ such that*

$$scal_{g_i(t)}(x) \leq \frac{C_D}{t} \quad \textit{for all} \quad i \geq 1, \quad x \in B_{g_i(t)}(p_0, D) \quad \textit{and} \qquad t \in (0,\tau].$$

Proof. Assume not. Then we can find a constant $D_0 > 0$ so that there exists $i_k \geq 1$, $t_k \in (0,\tau)$ and $p_k \in B_k(p_0, D_0) = B_{g_k(t_k)}(p_0, D_0)$ which satisfies

$$scal_k(p_k) = scal_{g_k(t_k)}(p_k) > \frac{4^k}{t_k}. \tag{3.10}$$

Claim: We can find $\{\bar{p}_k\}$ such that it satisfies 3.10 and

$$scal_{g_k(t)}(p) \leq 8 scal_k(\bar{p}_k)$$

for all $p \in B_k(\bar{p}_k, \dfrac{k}{\sqrt{scal_k(\bar{p}_k)}})$, $t \in [t_k - \dfrac{k}{scal_k(\bar{p}_k)}, t_k]$ with $d_k(\bar{p}_k, p_0) \leq D_0 + 1$.

As $K^{\mathbb{C}}_{g_k(t)} \geq 0$, by using the trace Harnack inequality in [30]. In particular , it gives $\frac{\partial}{\partial t}(t \cdot scal_{g(t)}) \geq 0$. This yields for any $t \in \left[t_k - \dfrac{k}{scal_k(\bar{p}_k)}, t_k\right]$,

$$scal_{g_k(t)} \leq \frac{t_k}{t} scal_k \leq \frac{t_k}{t_k - k/scal_k(\bar{p}_k)} < 2scal_k$$

The last inequality follows from the fact that $scal_k(\bar{p}_k) > \dfrac{4^k}{t_k}$ implies $\dfrac{k}{scal_k(\bar{p}_k)} < \dfrac{t_k}{4}$. So it suffices to find $\bar{p}_k$ satisfying

1. $scal_k(\bar{p}_k) > \dfrac{4^k}{t_k}$,and
2. $scal_k(p) \leq 4scal_k(\bar{p}_k)$ for all $p \in B_k(\bar{p}_k, k/\sqrt{scal_k(\bar{p}_k)})$.

If p_k does not satisfy (2), one can find a $x_1 \in B_k(p_k, k/\sqrt{scal_k(p_k)})$ such that $scal_k(x_1) > 4scal_k(p_k)$. Check if (2) holds for $\bar{p}_k = x_1$, that is

$$scal_k(p) \leq 4scal_k(x_1) \quad \text{for all} \quad p \in B_k(x_1, k/\sqrt{scal_k(x_1)}).$$

In case this is not satisfied, we process inductively and obtain a sequence $\{x_i\}_{i\geq 2}$ such that

1. $scal_k(x_i) > 4scal_k(x_{i-1}) > ... > 4^{i-1}scal_k(x_1) > \dfrac{4^{i+k}}{t_k}$,and
2. $x_i \in B_k(x_{i-1}, k/\sqrt{scal_k(x_{i-1})})$

If this sequence is finite ,that is $\{x_i\}_{i\geq 1} = \{x_1, x_2, ...x_N\}$. Then we can take $\bar{p}_k = x_N$ which satisfies the required properties. We now claim that this sequence can only be finite. Suppose $\{x_i\}_{i\geq 1}$ is a infinite sequence .
Since $scal_k(x_i) > 4^i scal_k(p_k)$, we have

$$d_k(x_i, x_{i-1}) < \frac{k}{\sqrt{scal_k(x_{i-1})}} < \frac{k}{2^i\sqrt{scal_k(p_k)}} < \frac{k}{2^{k+i}}\sqrt{\tau}.$$

This implies that for any $k >> \tau$

$$d_k(x_i, p_0) < \sum_{j=1}^{i} d_k(x_j, x_{j-1}) + d_k(p_k, p_0) \leq D_0 + \frac{k}{2^k}\sqrt{\tau} \leq D_0 + 1. \tag{3.11}$$

By convention , we denote $x_0 = p_k$. So $\{x_i\}_{i\geq 1}$ lies on a compact ball which contradicts that $scal_k(x_i) \to +\infty$. So the sequence $\{x_i\}_{i\geq 1}$ must be finite.

Because of (3.11) , it follows that for any $r > 0$, $B_k(p_0, r) \subset B_k(\bar{p}_k, r + D_0 + 1)$. Thus for any $r \in [D_0 + 3/2, D_0 + 2]$,

$$\begin{aligned}\frac{vol_k(B_k(\bar{p}_k, r))}{r^n} &\geq \frac{vol_k(B_k(\bar{p}_k, r - D_0 - 1))}{r^n} \geq vol_k(B_k(p_0, 1))\left(\frac{r - D_0 - 1}{r}\right)^n \\ &\geq v_0\left[\frac{1}{2(D_0+2)}\right]^n = \tilde{v}_0 > 0\end{aligned}$$

By Bishop-Gromov inequality , it ensures that the inequality holds for smaller radius. i.e.

$$\frac{vol_k(B_k(\bar{p}_k, r))}{r^n} \geq \tilde{v}_0, \quad \forall r \in (0, D_0 + 2].$$

Now, we define the parabolic scaling of the metric.

$$\tilde{g}_k(s) \doteq Q_k g(t_k + sQ_k^{-1}), \text{ where } Q_k = scal_k(\bar{p}_k),$$

In addition,we have a lower bound for the volume ratio. More precisely , we have for all $0 < r \leq (D_0 + 2)\sqrt{Q_k}$,

$$\begin{aligned}\frac{vol_{\tilde{g}_k(0)}(B_{\tilde{g}_k(0)}(\bar{p}_k, r))}{r^n} &= \frac{1}{r^n}\int_{B_{\tilde{g}_k(0)}(\bar{p}_k, r))} \sqrt{det(\tilde{g}_k(0))}dx \\ &= \frac{(\sqrt{Q}_k)^n}{r^n}\int_{B_k(\bar{p}_k, r/\sqrt{Q_k})} \sqrt{det(g_k)}dx \\ &\geq \tilde{v}_0 > 0, \end{aligned} \tag{3.12}$$

On the other hand, we have curvature bound on the new metric around $\bar{p}_k$.

$$0 \leq scal_{\tilde{g}_k(s)} = \frac{1}{Q_k} scal_{g(t_k + sQ_k^{-1})} \leq 8, \tag{3.13}$$

on $B_{\tilde{g}_k(0)}(\bar{p}_k, k)$ for all $s \in [-k, 0]$.

Combining these results and use the Theorem (see appendix) by Cheeger, Gromov and Taylor. We may conclude that

$$\mathrm{inj}_{\tilde{g}_k(0)}(\bar{p}_k) \geq c(n, \tilde{v}_0) > 0. \tag{3.14}$$

Joining the estimate (3.13) and (3.14) together and apply Hamilton's compactness (see [25]) to the pointed sequence

$$(M_k, \tilde{g}_k(s), \bar{p}_k), s \in [-k, 0].$$

We obtain a subsequence converging in the smooth Cheeger-Gromov sense to a complete smooth limit solution of the Ricci flow

$$(M_\infty, g_\infty(t), p_\infty) \quad t \in (-\infty, 0].$$

First noted that the diameter with respect to $\tilde{g}_k(s)$ tends to infinity as $Q_k \to +\infty$. So it is non-compact. It is non-flat since $scal_{g_\infty(0)}(p_\infty) = 1$. Finally, it has bounded curvature because of (3.13) with $K^{\mathbb{C}}_{g_\infty(t)} \geq 0$.
Moreover , from (3.16) , we have

$$\frac{vol_{g_\infty(0)}(B_{g_\infty(0)}(p_\infty, r))}{r^n} \geq \tilde{v}_0 \qquad \text{for all} \quad r > 0.$$

which contradicts the result of Lemma (3.5.3). □

3.6 Proof of short time existence for the positively curved case.

As soon as we establish the a-priori estimate of curvature around the soul, we are able to prove the short time existence of Ricci flow on the whole manifold.

Theorem 3.6.1. *Let (M^n, g) be an open manifold with $K^{\mathbb{C}}_g > 0$. Then there exists $\tau > 0$ and a sequence of closed Ricci flows $(M_i, g_i(t), p_0)_{t \in [0,\tau]}$ with $K^{\mathbb{C}}_{g_i(t)} > 0$ which converge in the smooth Cheeger-Gromov sense to a complete limit solution of the Ricci flow $(M, g_\infty(t), p_0)$ for $t \in [0, \tau]$, with $g_\infty(0) = g$.*

Proof. Consider the sequence $(M_i, g_i(t))$ with $t \in (0, \tau]$, coming from Proposition 3.4.2. For any j fixed, consider $B_g(p_0, j) \subset M$.

By Proposition 3.5.4, we can find some constant $L_j > 0$ such that

$$|Rm_{g_i(t)}| \leq \frac{L_j}{t} \quad \text{on} \quad B_{g_i(t)}(p_0, 2j) \quad \text{for all} \quad t \in [0, \tau], \quad i \geq 1.$$

As $K^{\mathbb{C}}_{g_i(t)} \geq 0$, $B_{g_i(0)}(p_0, 2j) \subset B_{g_i(t)}(p_0, 2j)$ for all $t \geq 0$. We conclude that

$$|Rm_{g_i(t)}| \leq \frac{L_j}{t} \quad \text{on} \quad B_{g_i(0)}(p_0, 2j) \quad \text{for all} \quad t \in (0, \tau], \quad i \geq 1. \tag{3.15}$$

On the other hand, by the result of Proposition 3.3.2, there exists a collection of diffeomorphism $\phi_i : B_g(p_0, j) \to M_i$ onto its image, and $l \in \mathbb{N}$ such that

$$|{}^g\nabla^m(\phi_i^* g_i(0) - g)|_g \leq \frac{1}{4}, \quad \text{for} \quad i \geq l \quad, m = 0, 1, 2 \quad \text{on} \quad B_g(p_0, j) \tag{3.16}$$

When $m = 0$, if we choose a coordinate at $p \in U$ such that, $g_{ab}(p) = \delta_{ab}$ and $\phi_i^* g_i(0)_{ab}(p) = \lambda_a \delta_{ab}$, then (3.16) implies

$$(\lambda_a - 1)^2 \leq \frac{1}{4}, \text{on} \quad B_g(p_0, j) \quad \text{for} \quad i \geq l$$

which implies

$$\frac{3}{4}\, g \leq \phi_i^* g_i(0) \leq \frac{5}{4}\, g \quad \text{on} \quad B_g(p_0, j) \quad , \text{for all} \quad i \geq l. \tag{3.17}$$

We now claim that under (3.16), indeed we have $\phi_i(B_g(p_0, j)) \subset B_{g_i(0)}(p_0, 2j)$ for all $i \geq l$.

Let $x \in B_g(p_0, j)$ and $\gamma : [0, 1] \to U$ be the minimal geodesic from p_0 to x.

$\phi_i \circ \gamma : [0,1] \to M_i$ is a curve from $\phi_i(p_0) = p_0$ to $\phi_i(x)$.

$$\begin{aligned} L(\phi_i \circ \gamma) &= \int_0^1 \sqrt{g_i(0)(d\phi_i(\gamma'), d\phi_i(\gamma'))}ds \\ &= \int_0^1 \sqrt{\phi_i^* g_i(0)(\gamma', \gamma')}ds \\ &\le \sqrt{\frac{5}{4}} \int_0^1 \sqrt{g(\gamma', \gamma')}ds \le 2j. \end{aligned}$$

It implies $d_{g_i(0)}(p_0, \phi_i(x)) \le 2j$.

Now, we estimate the curvature of $g_i(0)$ around the soul point p_0. For simplicity, we denote $\phi_i^* g_i(0)$ by h in the following steps.

$$|Rm(h)_{ijkl} - Rm(g)_{ijkl}|_g \le C(n)[|^g\nabla^g\nabla h|_g + |^g\nabla h|_g^2] \le \tilde{C}(n)$$

So, by (3.16) and uniformly equivalent of norm as stated above,

$$\begin{aligned} |Rm(\phi_i^* g_i(0))|_{\phi_i^* g_i(0)} = |Rm(h)|_h &\le C'(n)|Rm(h)|_g \\ &\le C'[|Rm(h) - Rm(g)|_g + |Rm(g)|_g] \\ &\le C''(n, g, j) \quad \text{for} \quad i \ge l \quad \text{on} \quad B_g(p_0, j) \end{aligned}$$

Noticed that the above constant C'' depends on n and $\sup\{|Rm(g)| : x \in B_g(p_0, j)\}$ only. In particular, we can find $r > 0$ small enough such that

$$|Rm(g_i(0))|_{g_i(0)} \le r^{-2} \quad \text{for} \quad i \ge l \quad \text{on} \quad \phi_i(B_g(p_0, j)). \tag{3.18}$$

Applying Corollary (3.2) in [2] to each $B_{g_i(0)}(p, r)$ where $p \in \phi_i(B_g(p_0, j))$. We deduce that there exists $\tilde{L}_j > 0$ such that

$$|Rm_{g_i(t)}| \le \tilde{L}_j \quad \text{on} \quad \phi_i(B_g(p_0, j)) \quad \text{for all} \quad t \in [0, \tau], \quad i \ge l.$$

Here $\tilde{L}_j$ depends on C'' and L_j only. After pulling back to $B_g(p_0, j)$ through ϕ_i, we get

$$|Rm(\phi_i^* g_i(t))| \le \tilde{L}_j \quad \text{on} \quad B_g(p_0, j) \quad \text{for all} \quad t \in [0, \tau], \quad i \ge l.$$

Combining this with extension of Shi-estimates, see [21].we reach furthermore

$$|\nabla^k Rm(\phi_i^* g_i(t))| \leq \tilde{L}_{j,k} \quad \text{on} \quad B_g(p_0, j) \quad \text{for all} \quad t \in [0, \tau], \quad i \geq l.$$

Here the connection and norm are calculated with respect to the metric $\phi_i^* g_i(t)$.

It remains to prove that the metrics $\phi_i^* g_i(t)$ on $B_g(p_0, j)$ have all space and time derivatives uniformly bounded.

Suppose it is true, one can apply Arzelà-Ascoli-Theorem to deduce that after passing to a subsequence, $\phi_i^* g_i(t)$ converges to $g_\infty(t)$ in the C^∞ topology on $B_g(p_0, j) \times [0, \tau] \subset M \times \mathbb{R}$. Doing this for each $j \in \mathbb{N}$ and apply the usual diagonal sequence argument, we can obtain a limit metric $g_\infty(t)$ which is a solution of the Ricci flow on M with initial metric $g_\infty(0) = g$.

Because of the equation of Ricci flow, it suffices to show that the metrics $\phi_i^* g_i(t)$ on $B_g(p_0, j)$ have all space derivatives uniformly bounded.
For simplicity, we denote $\phi_i^* g_i(t)$ by $g(t)$ in the following steps. We will illustrate the case of 1st order derivative. The higher order case is similar. Noted that the curvature of $g(t)$ is bounded by a constant $\tilde{L}_j$ on $B_g(p_0, j)$. The metrics $g(t), t \in [0, \tau]$ are uniformly equivalent to g on $B_g(p_0, j)$.
In general, we have

$$\tilde{\nabla} Rm = \nabla Rm + (\tilde{\nabla} - \nabla) * Rm = \nabla Rm + g(t)^{-1} * \tilde{\nabla} g(t) * Rm$$

where $\tilde{\nabla}$ denotes the connection induced by the metric g.
Combining this with the above estimates, we conclude that

$$|\tilde{\nabla} Rm|_g \leq c(n, j) + c'(n, j) |\tilde{\nabla} g(t)|_g$$

Thus, from the Ricci flow equation and above equality,

$$\begin{aligned}\frac{\partial}{\partial t}|\tilde{\nabla} g(t)|^2 &= 2\langle \tilde{\nabla} g(t), -2\tilde{\nabla} Ric(g(t))\rangle \\ &\leq C(n)|\tilde{\nabla} g(t)||\tilde{\nabla} Rm(g(t))| \\ &\leq C_1(n,j)|\tilde{\nabla} g(t)| + C_2(n,j)|\tilde{\nabla} g(t)|^2 \\ &\leq C_3(n,j)|\tilde{\nabla} g(t)|^2 + C_4(n,j)\end{aligned}$$

Here, the norm is calculated with respect to the metric g.

So $|\tilde{\nabla}\phi_i^* g_i(t)|_g$ is bounded by a constant depending on n, j and τ but independent of i. The higher order derivatives cases are similar. □

The completeness follows from the next Lemma.

Lemma 3.6.2. *There exists $L' > 0$ such that for all $r > 0$ and $t \in [0,\tau]$,*

$$B_{g_\infty(t)}(p_0, r) \subset B_{g_\infty(0)}(p_0, r + L't).$$

Proof. By theorem 3.1 in [2], there exists L independent of i such that

$$|Ric(g_i(t))|(x,t) \leq L \qquad ,\forall t \in [0,\tau],\ x \in B_{g_i(t)}(p_0, 1).$$

Let $q \in M$, $t \in [0,\tau]$. Let $c(s)$ be a minimal unit speed geodesic in $(M_i, g_i(t)$ from p_0 to q. If q is a conjugate point of p, we can consider q_n on $c(s)$ which converge to q. So without loss of generality, we assume q is not a conjugate point of p. We now estimate the left derivative of $r_i(t) = d_{g_i(t)}(p_0, q)$ as follows.

$$\frac{d}{dt} r_i(t) \geq -\int_0^{r_i(t)} Ric_{g_i(t)}(c'(s), c'(s))ds$$

If $r_i(t) \leq 1$, then $\frac{d}{dt} r_i(t) \geq -L$. Otherwise, let $\{e_i\}_{i=1}^n$ be a orthonormal vector at q where $e_1 = c'(r_i(t))$. We extend $\{e_j\}$ along $c(s)$ to $\{E_j\}$ by parallel translation. Let $\{V_j\}$ be Jocabi field along $c(s), s \in [0,1]$ where $V_j(0) = 0,\ V_j(1) = E_j(1)$. Let $Z_j(s)$ be a vector field along $c(s)$ where

$$Z_j(s) = \begin{cases} V_j(s) & \text{if } s \in [0,1] \\ E_j(s) & \text{if } s \in [1, r_i(t)]. \end{cases}$$

Then,

$$\begin{aligned}
-\int_1^{r_i(t)} Ric_{g_i(t)}(c'(s),c'(s))ds &= \sum_{j=1}^{n}\int_1^{r_i(t)} |\nabla_{c'}E_j|^2 - Ric_{g_i(t)}(E_1,E_j,E_j,E_1) \\
&= \sum_{j=1}^{n} I_r(Z_j,Z_j) - \sum_{j=1}^{n} I_1(Z_j,Z_j)
\end{aligned}$$

where $I_r(X,X)$, $I_1(X,X)$ are the index form of vector field X along the geodesic $c(s)$ and $c|_{[0,1]}(s)$ respectively. We estimate two terms separately. Since $Z_j(0)=0$ and $Z_j(r_i(t)) \perp e_j$, by index lemma, we have

$$I_r(Z_j,Z_j) \geq I_r(J_j,J_j)$$

where $J_j(s), s \in [0,r_i(t)]$ is the Jocabi field at which $J_j(0)=0, J_j(r_i(t))=e_j$. Hence, by second variational formula with the fact that $c(s)$ is minimal, we conclude that

$$I_r(Z_j,Z_j) \geq I_r(J_j,J_j) \geq 0.$$

On the other hand,

$$\sum_{j=1}^{n} I_1(Z_j,Z_j) = \sum_{j=1}^{n} I_1(V_j,V_j) = \Delta_{g_i(t)} d_{g_i(t)}.$$

By Laplacian comparison theorem and the fact that $K_{g_i(t)} \geq 0$,

$$\Delta_{g_i(t)} d_{g_i(t)} \leq \Delta d(1) = n-1.$$

Thus, we have

$$\frac{d}{dt} r_i(t) \geq -L-(n-1) = -L'.$$

Consequently, we can conclude that for all $q \in M$,

$$d_{g_i(0)}(p_0,q) \;\geq\; d_{g_i(t)}(p_0,q) \;\geq\; d_{g_i(0)}(p_0,q) - L't$$

where L' is independent of i and t. And this implies for all $R>0$, $i \in \mathbb{N}$,

$$B_{g_i(0)}(p_0,R) \subset B_{g_i(t)}(p_0,R) \subset B_{g_i(0)}(p_0,R+L't).$$

Taking $i \to \infty$, we yield the result.

Furthermore, let $\{p_n\}_{n=1}^{\infty}$ be a cauchy sequence in $(M, g_\infty(t)), t \in [0, \tau]$. There exists $R > 0$ such that

$$p_n \in B_{g_\infty(t)}(p_0, R) \subset B_{g_\infty(0)}(p_0, R + L't)$$

which implies that there exists $p \in (M, g_\infty(0))$ such that $d_{g_\infty(0)}(p_n, p) \to 0$. Thus,

$$d_{g_\infty(t)}(p_n, p) \leq d_{g_\infty(0)}(p_n, p) \to 0.$$

Thus $(M, g_\infty(t))$ is complete for all $t \in [0, \tau]$. $\square$

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